Complex analysis, homework 11, solutions.

Exercise 1. [9 points] Give three Laurent expansions in powers of z for the function

$$f(z) = \frac{i-3}{(z-i)(z-3)} = \frac{1}{z-i} - \frac{1}{z-3}$$

and specify the annular domains in which those expansions are valid.

Solution. The function f is analytic on $\mathbb{C} \setminus \{i, 3\}$. So it is analytic on the following annular domains centered at 0

$$D_1 = \{z : |z| < 1\}, \qquad D_2 = \{z : 1 < |z| < 3\}, \qquad D_3 = \{z : 3 < |z|\}.$$

Note that in D_1 we can actually include 0, because f is analytic at 0, so we actually know that f can be expanded as a Taylor series on D_1.

If $|z| < 1$, then $|z/i| < 1$ and therefore

$$\frac{1}{z-i} = \frac{1}{(-i)} \cdot \frac{1}{1-(z/i)} = -\frac{1}{i}\sum_{n=0}^{\infty}(z/i)^n = -\sum_{n=0}^{\infty}\frac{z^n}{i^{n+1}}.$$

On the other hand, if $|z| > 1$, then $|i/z| < 1$ and therefore

$$\frac{1}{z-i} = \frac{1}{z} \cdot \frac{1}{1-(i/z)} = \frac{1}{z}\sum_{n=0}^{\infty}(i/z)^n = \sum_{n=0}^{\infty}\frac{i^n}{z^{n+1}} = \sum_{n=1}^{\infty}\frac{i^{n-1}}{z^n}.$$

We proceed similarly to get

$$\frac{1}{z-3} = -\sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}} \qquad \text{if } |z| < 3,$$

$$\frac{1}{z-3} = \sum_{n=1}^{\infty}\frac{3^{n-1}}{z^n} \qquad \text{if } |z| > 3.$$

Combining this we get

$$f(z) = \sum_{n=0}^{\infty}\left(\frac{1}{3^{n+1}} - \frac{1}{i^{n+1}}\right)z^n \qquad \text{if } z \in D_1,$$

$$f(z) = \sum_{n=0}^{\infty}\frac{1}{3^{n+1}}z^n + \sum_{n=1}^{\infty}\frac{i^{n-1}}{z^n} \qquad \text{if } z \in D_1,$$

$$f(z) = \sum_{n=1}^{\infty}\left(i^{n-1} - 3^{n-1}\right)\frac{1}{z^n} \qquad \text{if } z \in D_3.$$

Exercise 2. [12 points] Find the radius of convergence of the following power series. Explain your answer.

(1) $\sum_{n=0}^{\infty}(n+1)^{2n}z^n$; (2) $\sum_{n=0}^{\infty}\left(n2^n + 3^n\right)z^n$;

(3) $\sum_{n=0}^{\infty}(\rho e^{i\theta}z)^n$, for some $\theta \in \mathbb{R}$ and $\rho > 0$;

(4) $\sum_{n=0}^{\infty} \frac{nz^{2n}}{(4i)^n}$;

Solution.

(1) We have $a_n = (n+1)^{2n}$. Let $r > 0$. Then $|a_n r^n| = ((n+1)^2 r)^n \geq 2^n$ for $n \geq \sqrt{2/r}$. Since $2^n \to +\infty$ as $n \to +\infty$, by comparison we get that $|a_n r^n| \to +\infty$. In particular $(a_n r^n)$ is not a bounded sequence for any $r > 0$. So the radius of convergence is 0.

(2) We have $a_n = (n2^n + 3^n)$.
- Let $r > 1/3$. Then $|a_n r^n| = n(2r)^n + (3r)^n \geq (3r)^n + \infty$ as $n \to +\infty$ because $3r > 1$. So the sequence $(a_n r^n)$ is not bounded for any $r > 1/3$.
- Let $r < 1/3$. Then $(3r)^n \to 0$ as $n \to +\infty$ because $3r < 1$. Moreover $n(2r)^n \to 0$ as $n \to +\infty$ because $2r < 1$ and geometric sequences "dominate" polynomial sequences. Therefore $|a_n r^n| = n(2r)^n + (3r)^n \to 0$ as $n \to +\infty$. So the sequence $(a_n r^n)$ is bounded for any $r < 1/3$.

So the radius of convergence is 1/3.

Note that we don't need to study the behavior of the sequence when $r = 1/3$ to conclude.

(3) We have $a_n = (\rho e^{i\theta})^n$. Therefore $|a_n r^n| = (\rho r)^n$ so the sequence $(a_n r^n)$ is bounded if and only if $r \leq 1/\rho$. So the radius of convergence is $1/\rho$.

(4) We have $a_{2n} = \frac{n}{(4i)^n}$ and $a_{2n+1} = 0$ for $n \geq 0$. The sequence $(a_{2n+1}r^{2n+1})$ is bounded for any $r \geq 0$ (because it is constant equal to 0). So the sequence $(a_n r^n)$ is bounded whenever the sequence $(a_{2n}r^{2n})$ is bounded. But we have $|a_{2n}r^{2n}| = n(r/2)^{2n}$. If $r < 2$, this sequence converges to 0 and therefore is bounded. If $r > 2$, this sequence tends to $+\infty$ and therefore is not bounded. So the radius of convergence is 2.

Exercise 3. [5 points] Show that the following function is entire

$$f(z) = \begin{cases} \dfrac{\sin(z)}{z-\pi} & \text{if } z \neq \pi, \\ -1 & \text{if } z = \pi. \end{cases}$$

Solution. The function sin is analytic on $\mathbb{C}$, so by Taylor's theorem, it is equal to its Taylor series at π on the whole complex plane:

$$\sin(z) = \sum_{n=0}^{\infty} \frac{\sin^{(n)}(\pi)}{n!}(z-\pi)^n, \qquad z \in \mathbb{C}.$$

Note that $\sin(\pi) = 0$ so the first term is 0. Therefore, for any $z \neq 0$,

$$f(z) = \frac{1}{z-\pi}\sum_{n=1}^{\infty} \frac{\sin^{(n)}(\pi)}{n!}(z-\pi)^n = \sum_{k=0}^{\infty} \frac{\sin^{(k+1)}(\pi)}{(k+1)!}(z-\pi)^k.$$

Since, $\sin'(\pi) = \cos(\pi) = -1$, this last power series equals $-1 = f(\pi)$ when $z = \pi$. So we conclude that, for any $z \in \mathbb{C}$,

$$f(z) = \sum_{k=0}^{\infty} \frac{\sin^{(k+1)}(\pi)}{(k+1)!}(z-\pi)^k.$$

In particular, by proving this formula, we showed that this last series is convergent for any $z \in \mathbb{C}$ that is its radius of convergence is infinite. By the theorem of Sec. 71, this power series is analytic on $\mathbb{C}$ and therefore f is entire.

Note that we did not need to calculate all the coefficients of the series (even if we could have). This means that we can replace $\sin(z)$ *by any entire function* $g(z)$ *such that* $g(\pi) = 0$ *and* $g'(\pi) = -1$ *and the result would have been true as well.*

Exercise 4. [4 points] Recall $\operatorname{Log} z = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^n$, for $|z-1| < 1$. For $|z-1| < 1$, let C_z be a contour from 1 to z included in the open disk centered at 1 with radius 1. Write the following quantity as a power series in z around 1:

$$\int_C \operatorname{Log}(w)\,\mathrm{d}w.$$

Justify your answer.

Solution. We have, for $|z-1| < 1$,

$$\operatorname{Log} z = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^n.$$

Since this power series is convergent on the whole open disk centered at 1 with radius 1 and the contour C_z is included in this disk, we can integrate it term by term (by the theorem of Sec. 71) and we get

$$\int_{C_z} \operatorname{Log}(w)\,\mathrm{d}w = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int_{C_z} (w-1)^n\,\mathrm{d}w.$$

Now, note that $(w-1)^n$ is continuous and has antiderivative $\frac{(w-1)^{n+1}}{n+1}$ on the whole complex plane, so we have

$$\int_{C_z} (w-1)^n\,\mathrm{d}w = \left[\frac{(w-1)^{n+1}}{n+1}\right]_1^z = \frac{(z-1)^{n+1}}{n+1} - 0.$$

Therefore, we get

$$\int_{C_z} \operatorname{Log}(w)\,\mathrm{d}w = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)}(z-1)^{n+1} = \sum_{k=2}^{\infty} \frac{(-1)^k}{(k-1)k}(z-1)^k.$$