Complex analysis, homework 1 plus 2, due February 1st.

Exercise 1.[12 points] Compute the following quantities. Show your steps.

(1) $(3-i)(-2+5i)-3+2i$
(2) $\dfrac{-3+2i}{2-i}$
(3) $(1+i)^3$

Exercise 2. [4 points] Which of the points $z_1 = 3+6i$ and $z_2 = 5-4i$ is closer to the origin?

Exercise 3. [6 points]

(1) Show that, for any $z \in \mathbb{C}$, $z^2+1 = (z-i)(z+i)$.
(2) Prove that the equation $z^2+1=0$ has exactly two solutions, which are i and $-i$.

Exercise 4. [4 points] Sketch the region in the complex plane $\{z \in \mathbb{C} : |z-2+i| \leq 3\}$, that is the set of all points z such that $|z-2+i| \leq 3$.

Exercise 5. [4 points] Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be complex numbers. Express $\mathrm{Re}(z_1\overline{z_2})$ in terms of x_1, x_2, y_1, y_2. What does it represent for the vectors z_1 and z_2?

Exercise 6. [4 points] Let $z_1, z_2 \in \mathbb{C}$ be in the upper left quarter plane (that is with negative real part and positive imaginary part). Prove that

$$\mathrm{Arg}(z_1 z_2) = \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) - 2\pi.$$

Exercise 7. [4 points] Let $w, z \in \mathbb{C}$ with $|w| = 1$ and $z \neq w$. Prove that

$$\left|\frac{w-z}{1-\overline{w}z}\right| = 1.$$

Exercise 8. [4 points] Suppose that $z_1 z_2$ is real and non-zero. Prove that there exists a real number c such that $z_1 = c\bar{z}_2$.

Exercise 9. [4 points] Prove that for any z with modulus $R > 1$, one has

$$\left|\frac{z^4+\mathrm{i}z}{z^2+z+1}\right| \leq \frac{R^4+R}{(R-1)^2}.$$