Complex analysis, homework 3, solutions.

Exercise 1.[4 points] Calculate $(-2+2i)^{10}$. Give your result in the form $x+iy$ with x and y real numbers. Show you steps.
Remark: We have seen a method in class for this, do not expand directly $(-2+2i)^{10}$.

Solution. Let $z=-2+2i$. Then $|z|=\sqrt{(-2)^2+2^2}=\sqrt{8}=2\sqrt{2}$. Then, we have

$$z=2\sqrt{2}\left(-\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)=2\sqrt{2}\left(\cos(3\pi/4)+i\sin(3\pi/4)\right)=2\sqrt{2}e^{i3\pi/4}.$$

Therefore,

$$(-2+2i)^{10}=\left(2\sqrt{2}e^{i3\pi/4}\right)^{10}=\left(2\sqrt{2}\right)^{10}e^{i10\cdot\frac{3\pi}{4}}=2^{15}e^{i\frac{3\pi}{2}+2i\pi\cdot 3}=32768\cdot(-i)=-32768i.$$

Exercise 2.[6 points]

(1) Find the fourth roots of i. Give them in exponential forms and then represent them on a picture. Highlight the principal fourth root.
(2) Find the third roots of $-8+8\sqrt{3}i$? Give them in exponential forms and then represent them on a picture. Highlight the principal third root.

Solution.

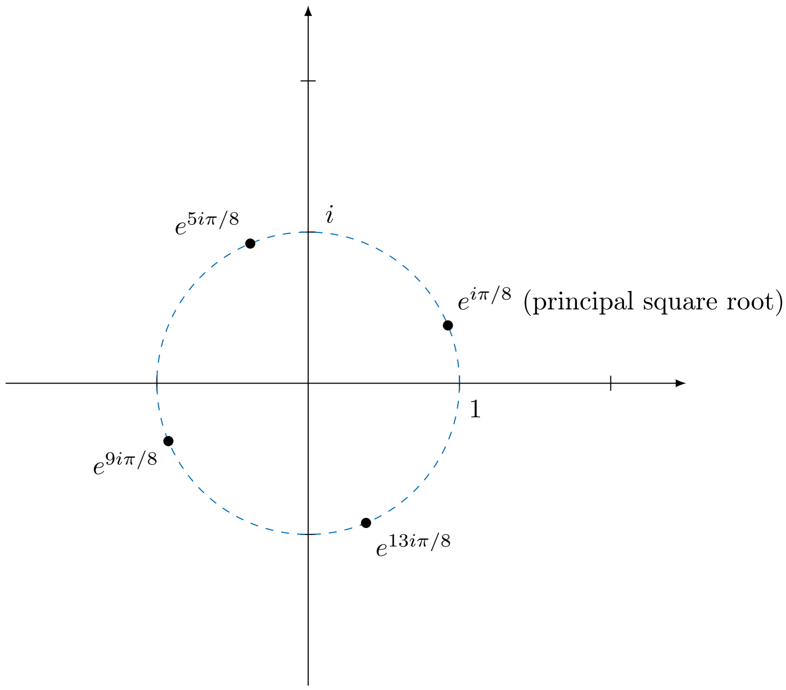
(1) We have $i=e^{i\frac{\pi}{2}}$. So applying the result seen in class, we know that the fourth roots of i are

$$\exp\left(i\frac{\frac{\pi}{2}+0}{4}\right),\exp\left(i\frac{\frac{\pi}{2}+2\pi}{4}\right),\exp\left(i\frac{\frac{\pi}{2}+4\pi}{4}\right),\exp\left(i\frac{\frac{\pi}{2}+6\pi}{4}\right),$$

and they can be rewritten as

$$\exp\left(i\frac{\pi}{8}\right),\exp\left(i\frac{5\pi}{8}\right),\exp\left(i\frac{9\pi}{8}\right),\exp\left(i\frac{13\pi}{8}\right).$$

Since $\mathrm{Arg}(i)=\frac{\pi}{2}$, the principal fourth root is $e^{i\,\mathrm{Arg}(i)/4}=e^{i\pi/8}$.



(2) Let $z = -8 + 8\sqrt{3}i$. We have $|z| = \sqrt{(-8)^2 + (8\sqrt{3})^2} = \sqrt{8^2 + 3 \cdot 8^2} = \sqrt{4 \cdot 8^2} = 16$. Hence, we write

$$z = 16\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 16\exp\left(i\frac{2\pi}{3}\right).$$

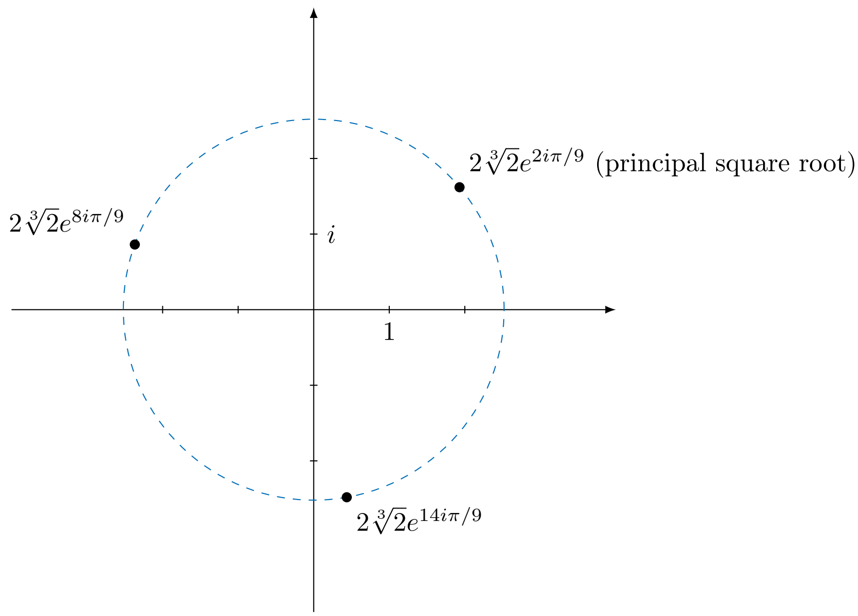
So applying the result seen in class, we know that the third roots of z are

$$16^{1/3}\exp\left(i\frac{\frac{2\pi}{3}+0}{3}\right), 16^{1/3}\exp\left(i\frac{\frac{2\pi}{3}+2\pi}{3}\right), 16^{1/3}\exp\left(i\frac{\frac{2\pi}{3}+4\pi}{3}\right),$$

and they can be rewritten as

$$2\sqrt[3]{2}\exp\left(i\frac{2\pi}{9}\right), 2\sqrt[3]{2}\exp\left(i\frac{8\pi}{9}\right), 2\sqrt[3]{2}\exp\left(i\frac{14\pi}{9}\right).$$

Since $\operatorname{Arg}(z) = \frac{2\pi}{3}$, the principal third root is $|z|^{1/3}e^{i\operatorname{Arg}(z)/3} = 2\sqrt[3]{2}e^{2i\pi/9}$.

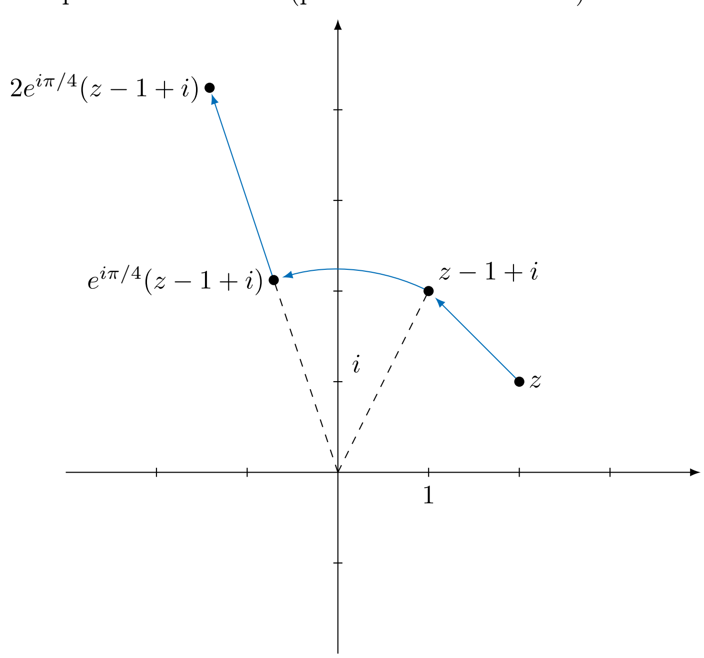


Exercise 3.[4 points] We consider the following transformation $z \mapsto 2e^{i\pi/4}(z-1+i)$. Describe its effect on a point z of the complex plane in words (there should be three successive simple steps). Illustrate it with a picture in the case $z = 2+i$ (that is represent z and $2e^{i\pi/4}(z-1+i)$, as well as the results of the successive steps described earlier).

Solution. The effects of the transformation are successively

- translation by the vector $-1+i$;
- rotation centered at 0 with angle $\pi/4$;
- scaling centered at 0 with factor 2.

Note that the two last steps can be switched (product is commutative).



Exercise 4.[4 points] Prove that $\lim_{z\to 1-i} \dfrac{2z+1}{iz+1}$ exists and give its value in the form $x+iy$.

Solution. We want to prove

$$\lim_{z\to 1-i}\frac{2z+1}{iz+1}=\frac{2(1-i)+1}{i(1-i)+1},$$

which equals $\frac{8}{5}-\frac{1}{5}i$ by the previous calculation. Let $\varepsilon>0$. We choose $\delta=\min(\varepsilon,\sqrt{5}-1)>0$. We set $z_0=1-i$ and consider z such that $0<|z-z_0|<\delta$, then we have

$$\begin{aligned}\left|\frac{2z+1}{iz+1}-\frac{2z_0+1}{iz_0+1}\right| &= \left|\frac{(2z+1)(iz_0+1)-(2z_0+1)(iz+1)}{(iz+1)(iz_0+1)}\right|\\ &= \left|\frac{2izz_0+iz_0+2z+1-2izz_0-iz-2z_0-1}{(iz+1)(iz_0+1)}\right|\\ &= \left|\frac{i(z_0-z)+2(z-z_0)}{(iz+1)(iz_0+1)}\right|\\ &= \frac{|2+i|\cdot|z_0-z|}{|iz+1|\cdot|iz_0+1|}\\ &= \frac{|z_0-z|}{|iz+1|},\end{aligned}$$

where in the last equality we used that $|iz_0+1|=|2+i|$, since $z_0=1-i$. We then use that $|z-z_0|<\delta$ to get

$$\left|\frac{2z+1}{iz+1}-\frac{2z_0+1}{iz_0+1}\right|<\frac{\delta}{|iz+1|}.$$

On the other hand,

$$\begin{aligned}|iz+1| &= |iz_0+1+i(z-z_0)|\\ &\geq |iz_0+1|-|i(z-z_0)| && \text{(triangle inequality)}\\ &= \sqrt{5}-|z-z_0| && (|iz_0+1|=|2+i|=\sqrt{5})\\ &> \sqrt{5}-\delta && (|z-z_0|<\delta)\\ &\geq 1 && (\delta\leq\sqrt{5}-1).\end{aligned}$$

Hence, we get

$$\left|\frac{2z+1}{iz+1}-\frac{2z_0+1}{iz_0+1}\right|<\delta\leq\varepsilon.$$

This proves that

$$\lim_{z\to 1-i}\frac{2z+1}{iz+1}=\frac{2(1-i)+1}{i(1-i)+1}=\frac{4}{5}-\frac{7}{5}i.$$

Exercise 5.[5 points] Let f be a function defined on $\mathbb{C}$. We say that f is Lipschitz on $\mathbb{C}$ if there exists $K>0$ such that, for any $z,z'\in\mathbb{C}$,

$$|f(z)-f(z')|\leq K|z-z'|.$$

Prove that, if f is Lipschitz on $\mathbb{C}$, then f has a limit at any point in $\mathbb{C}$.

Solution. Let z_0 in $\mathbb{C}$, we will prove that f is continuous at z_0. Let $\varepsilon>0$. We set $\delta=\varepsilon/K>0$ and consider consider z such that $|z-z_0|<\delta$. Then we have

$$|f(z)-f(z_0)|\leq K|z-z_0|<K\delta=\varepsilon.$$

Therefore, $\lim_{z\to z_0}f(z)=f(z_0)$. This proves that f is continuous at z_0 and therefore on $\mathbb{C}$.

Exercise 6.[5 points] Prove that $\lim\limits_{z\to -1}\mathrm{Arg}(z)$ does not exist.

Solution. Note that if $\mathrm{Re}(z) < 0$ and $\mathrm{Im}(z) \geq 0$, then $\mathrm{Arg}(z) \in (\pi/2, \pi]$, and if $\mathrm{Re}(z) < 0$ and $\mathrm{Im}(z) < 0$, then $\mathrm{Arg}(z) \in (-\pi, -\pi/2)$. So $\mathrm{Arg}(z)$ takes values both in $(\pi/2, \pi]$ and in $(-\pi, -\pi/2)$ arbitrarily close to -1 so it cannot have a limit.

Let's prove it properly. For the sake of contradiction, assume $\lim_{z\to -1} \mathrm{Arg}(z) = w$ for some $w \in \mathbb{C}$. Let $\varepsilon = \pi/2 > 0$. Then there is a $\delta > 0$ such that, for any z with $0 < |z+1| < \delta$, $|\mathrm{Arg}(z) - w| < \varepsilon = \pi/2$. Consider

$$z_1 = -1 + i\frac{\delta}{2} \qquad \text{and} \qquad z_2 = -1 - i\frac{\delta}{2}.$$

We have, for $k = 1$ or 2, $0 < |z_k + 1| < \delta$, so $|\mathrm{Arg}(z_k) - w| < \pi/2$. Hence, we get, using the triangle inequality,

$$|\mathrm{Arg}(z_1) - \mathrm{Arg}(z_2)| \leq |\mathrm{Arg}(z_1) - w| + |w - \mathrm{Arg}(z_2)| < \pi.$$

But, on the other hand, we have $\mathrm{Arg}(z_1) \in (\pi/2, \pi]$ and $\mathrm{Arg}(z_2) =\in (-\pi, -\pi/2)$, so

$$|\mathrm{Arg}(z_1) - \mathrm{Arg}(z_2)| \geq \mathrm{Arg}(z_1) - \mathrm{Arg}(z_2) > \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) > \pi.$$

This is our contradiction.

Exercise 7.[8 points] Let $z_0 \in \mathbb{C}$. **Prove or disprove** the following statements:

(1) Let f and g be functions defined on a deleted neighborhood of z_0.

$$\text{If } \lim_{z\to z_0} f(z) = \infty \text{ and } \lim_{z\to z_0} g(z) = \infty, \text{ then } \lim_{z\to z_0} (f(z) + g(z)) = \infty.$$

(2) Let f and g be functions defined on a deleted neighborhood of z_0.

$$\text{If } \lim_{z\to z_0} f(z) = \infty \text{ and } \lim_{z\to z_0} g(z) = \infty, \text{ then } \lim_{z\to z_0} (f(z) \times g(z)) = \infty.$$

Remark: In order to disprove a result, you have to give a counterexample.

Solution.

(1) This is false. Consider the functions

$$f(z) = \frac{1}{z - z_0} \qquad \text{and} \qquad g(z) = \frac{1}{z_0 - z},$$

defined for any $z \neq z_0$. Then, we have

$$\lim_{z\to z_0} f(z) = \infty \text{ and } \lim_{z\to z_0} g(z) = \infty,$$

as a consequence of the fact that $\lim_{z\to z_0} 1/f(z) = \lim_{z\to z_0}(z - z_0) = 0$ combined with the theorem seen in class for limits involving infinity (and similarly for g). But, on the other hand, $f(z) + g(z) = 0$ for any $z \neq z_0$, so we have

$$\lim_{z\to z_0} (f(z) + g(z)) = 0 \neq \infty.$$

(2) This is true. Let f and g be functions defined on a deleted neighborhood of z_0 such that

$$\lim_{z\to z_0} f(z) = \infty \text{ and } \lim_{z\to z_0} g(z) = \infty.$$

We want to prove that $\lim_{z\to z_0} f(z)g(z) = \infty$.

Approach 1 (using the definition): Let $\varepsilon > 0$. We have $\sqrt{\varepsilon} > 0$.

Since $\lim_{z\to z_0} f(z) = \infty$, there is a $\delta_1 > 0$ such that for any z with $|z - z_0| < \delta_1$, we have $|f(z)| > 1/\sqrt{\varepsilon}$.

Since $\lim_{z\to z_0} g(z) = \infty$, there is a $\delta_2 > 0$ such that for any z with $|z - z_0| < \delta_2$, we have $|g(z)| > 1/\sqrt{\varepsilon}$.

Let $\delta = \min(\delta_1, \delta_2) > 0$. Consider z such that $|z - z_0| < \delta$. Then we have $|f(z)| > 1/\sqrt{\varepsilon}$ and $|g(z)| > 1/\sqrt{\varepsilon}$. Hence,

$$|f(z)g(z)| = |f(z)| \cdot |g(z)| > \frac{1}{\sqrt{\varepsilon}} \cdot \frac{1}{\sqrt{\varepsilon}} = \frac{1}{\varepsilon}.$$

This proves $\lim_{z\to z_0} f(z)g(z) = \infty$.

Approach 2 (using results): By the theorem concerning limits involving infinity, we have

$$\lim_{z\to z_0} \frac{1}{f(z)} = 0 \text{ and } \lim_{z\to z_0} \frac{1}{g(z)} = 0.$$

Therefore, using the result for products of limits (we can apply it to these finite limits!), we get

$$\lim_{z\to z_0} \frac{1}{f(z)g(z)} = \lim_{z\to z_0} \frac{1}{f(z)} \cdot \frac{1}{g(z)} = 0 \cdot 0 = 0.$$

Using again the theorem concerning limits involving infinity, this implies that $\lim_{z\to z_0} f(z)g(z) = \infty$.