Complex analysis, homework 4 due February 15th.

Exercise 1.[8 points] For the following functions, say at which points they are differentiable and find their derivatives. Show your steps.

(1) $f(z) = \dfrac{z^2}{iz+1}$
(2) $f(z) = z(z^2 + iz)^5$

Exercise 2.[5 points] Let $z_0 \in \mathbb{C}$. Let f be a function differentiable at z_0. For any $z \in \mathbb{C}$ such that $f(\overline{z})$ is defined, we set

$$g(z) = \overline{f(\overline{z})}.$$

Prove that g is differentiable at $\overline{z_0}$ and express $g'(\overline{z_0})$ in terms of $f'(z_0)$.

Exercise 3.[8 points] Let $f(z) = z\,\mathrm{Im}(z)$ for $z \in \mathbb{C}$. Find the points $z \in \mathbb{C}$ where f is differentiable and find its derivative $f'(z)$ at these points. For all the other points in the complex plane, prove that f is not differentiable at these points.

Exercise 4.[9 points] Let f be a function differentiable on $\mathbb{C}$.

(1) Prove that if $\mathrm{Re}(f)$ is constant on $\mathbb{C}$, then f is constant on $\mathbb{C}$.
(2) Prove that if $|f|$ is constant on $\mathbb{C}$, then f is constant on $\mathbb{C}$.

Hint: Use the Cauchy-Riemann equations. You can use the following fact: if a real-valued function on $\mathbb{R}^2$ has its both partial derivatives that are zero on $\mathbb{R}^2$, then this function is constant on $\mathbb{R}^2$. For (b), you can start by squaring the modulus and differentiate either with respect to x or with respect to y.