**Complex analysis, homework 6 due March 7th.**

**Exercise 1.**[7 points] Calculate the following quantities:

(1) $\sin\left(\frac{\pi}{3} + i\ln(2)\right)$;
(2) $(\sqrt{3}-i)^{2-i}$ and P.V.$(\sqrt{3}-i)^{2-i}$.

**Exercise 2.**[6 points] Evaluate the following integrals:

(1) $\displaystyle\int_0^1 t(2+it^2)^2\,\mathrm{d}t$;
(2) $\displaystyle\int_0^\pi \cos(2t+it)\,\mathrm{d}t$;

**Exercise 3.**[3 points] Find the zeros of the function $f$ defined by $f(z) = \cos(iz+1)$ for $z \in \mathbb{C}$.

**Exercise 4.**[4 points] Solve the equation P.V. $z^i = -e$.
*Hint:* Write $\operatorname{Log} z = a + ib$ and first solve for $a$ and $b$. Then recover $z$ from $a$ and $b$.

**Exercise 5.**[5 points] Let $I$ be a real interval and $w\colon I \to \mathbb{C}$ be a function. Assume $w$ is differentiable at some $t \in I$. Prove $|w|^2$ is differentiable at $t$ and find its derivative in terms of $w'(t)$.

**Exercise 6.**[5 points] Let $\alpha \in \mathbb{R}$. Consider the branch $F$ of the log defined by

$$F(z) = \ln r + i\theta \quad \text{for} \quad z = re^{i\theta} \quad \text{with} \quad r > 0 \quad \text{and} \quad \alpha < \theta < \alpha + 2\pi.$$

Let $D = \{re^{i\theta} : r > 0 \text{ and } \alpha < \theta < \alpha + 2\pi\}$. Recall $F$ is analytic on $D$ and $F'(z) = 1/z$ for any $z \in D$. Let $c \in \mathbb{C}$. We define $G(z) = e^{cF(z)}$ for any $z \in D$.

(1) Explain why $G$ is a branch of the power function $z^c$ on $D$, that is $G$ is analytic on $D$ and, for any $z \in D$, $G(z)$ is one of the values of $z^c$.
(2) For any $z \in D$, show that $G'(z) = ce^{(c-1)F(z)}$.

*Remark:* The principal value of the power function is only an arbitrary choice (as for the principal value of the log). Sometimes considering another one can be useful. Note that here the derivative depends on the branch chosen, which is not the case for the log.