## Complex analysis, homework 8, solutions

**Exercise 1.** [6 points] Let $C$ be the arc defined by

$$z(t) = \begin{cases} \pi e^{i\pi t} & \text{if } 0 \le t \le 1, \\ -\pi + i(t-1)\ln(2) & \text{if } 1 \le t \le 2, \end{cases}$$

and $f(z) = \cos(z)\sin^2(z)$. Calculate the following integral (give your answer in $x + iy$ form)

$$\int_C f(z)\,\mathrm{d}z.$$

**Solution.** $C$ is a contour because, at any $t \in [0,1)\cup(1,2]$, $z(t)$ is differentiable, $z'(t)$ is continuous and nonzero. The function $f$ has an antiderivative $F(z) = \frac{1}{3}(\sin(z))^3$ on the domain $\mathbb{C}$ and the contour $C$ lies entirely in $\mathbb{C}$ so by the theorem of Section 48,

$$\int_C f(z)\,\mathrm{d}z = F(z(2)) - F(z(0)) = F(-\pi + i\ln 2) - F(\pi).$$

Note that $F(\pi) = 0$ and

$$\sin(-\pi + i\ln 2) = \frac{e^{i(-\pi+i\ln 2)} - e^{-i(-\pi+i\ln 2)}}{2i} = \frac{e^{-i\pi}e^{-\ln 2} - e^{i\pi}e^{\ln 2}}{2i} = \frac{-\frac{1}{2} + 2}{2i} = -\frac{3i}{4}$$

so that $F(-\pi + i\ln 2) = \frac{9i}{64}$. Finally, we get

$$\int_C f(z)\,\mathrm{d}z = \frac{9i}{64}.$$

**Exercise 2.** [6 points] Let $z_0 \in \mathbb{C}$ and $r > 0$. Let $C$ be the positively oriented circle of radius $r$ about $z_0$ given by

$$z(\theta) = z_0 + re^{i\theta}, \quad 0 \le \theta \le 2\pi.$$

Evaluate the following integral (give your answer in terms of $z_0$)

$$\int_C \frac{z+i}{z-z_0}\,\mathrm{d}z.$$

**Solution.** Using the definition of contour integrals

$$\begin{aligned}\int_C \frac{z+i}{z-z_0}\,\mathrm{d}z &= \int_0^{2\pi} \frac{z_0 + re^{i\theta} + i}{z_0 + re^{i\theta} - z_0} rie^{i\theta}\,\mathrm{d}\theta \\ &= \int_0^{2\pi} \frac{z_0 + i + re^{i\theta}}{re^{i\theta}} rie^{i\theta}\,\mathrm{d}\theta \\ &= \int_0^{2\pi} (iz_0 - 1 + ire^{i\theta})\,\mathrm{d}\theta \\ &= 2\pi(iz_0 - 1) + \left[re^{i\theta}\right]_0^{2\pi} \\ &= 2\pi(iz_0 - 1).\end{aligned}$$

**Alternative approach:** With Cauchy integral formula applied with $f(z) = z + i$ which is analytic on and within $C$, we have

$$\int_C \frac{z+i}{z-z_0}\,\mathrm{d}z = 2i\pi f(z_0) = 2i\pi(z_0+i) = 2\pi(iz_0-1).$$

**Exercise 3.** [6 points] Let $C$ be a closed contour. Let $f$ be a piecewise continuous function on $C$. Prove that the integral $\int_C f(z)\,\mathrm{d}z$ does not depend of the choice of the initial point of the contour. More precisely, assume $C$ is given by $z = z(t)$, $a \le t \le b$, fix some $t_0 \in [a,b]$ and define $C'$ by

$$z = w(t) = \begin{cases} z(t) & \text{if } t_0 \le t \le b, \\ z(t-b+a) & \text{if } b \le t \le b-a+t_0, \end{cases}$$

Then you have to prove $\int_C f(z)\,\mathrm{d}z = \int_{C'} f(z)\,\mathrm{d}z$.

**Solution.** Using the definition of contour integrals

$$\begin{aligned}\int_{C'} f(z)\,\mathrm{d}z &= \int_{t_0}^{b-a+t_0} f(w(t))w'(t)\,\mathrm{d}t \\ &= \int_{t_0}^{b} f(z(t))z'(t)\,\mathrm{d}t + \int_{b}^{b-a+t_0} f(z(t-b+a))z'(t-b+a)\,\mathrm{d}t.\end{aligned}$$

In the second term we use the change of variable $s = t-b+a$, noting that when $t$ goes from $b$ to $b-a+t_0$, $s$ goes from $a$ to $t_0$. Hence, we get

$$\int_{C'} f(z)\,\mathrm{d}z = \int_{t_0}^{b} f(z(t))z'(t)\,\mathrm{d}t + \int_{a}^{t_0} f(z(s))z'(s)\,\mathrm{d}s = \int_{a}^{b} f(z(t))z'(t)\,\mathrm{d}t = \int_C f(z)\,\mathrm{d}z,$$

where we replaced $s$ by $t$ (it is just a dummy variable) and then combined both integrals.

**Exercise 4.** [6 points] Let $C$ be the arc defined by

$$z(t) = \begin{cases} it & \text{if } 0 \le t \le 1, \\ i + (t-1) & \text{if } 1 \le t \le 2, \\ 1 + i - i(t-2) & \text{if } 2 \le t \le 3, \\ 1 - (t-3) & \text{if } 3 \le t \le 4. \end{cases}$$

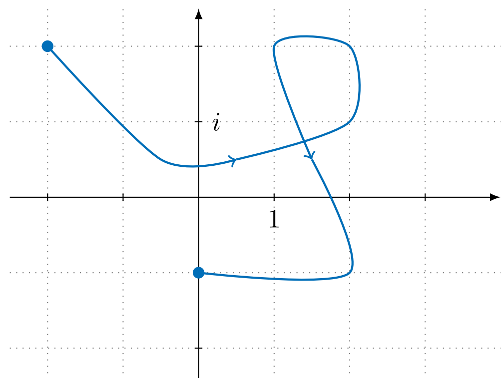
Evaluate the following integral (give your answer in $x + iy$ form)

$$\int_C \frac{e^{z^2}}{z^2+4}\,\mathrm{d}z$$

**Solution.** First note that $C$ is a contour because, at any $t \in [0,1) \cup (1,2) \cup (2,3) \cup (3,4]$, $z(t)$ is differentiable, $z'(t)$ is continuous and nonzero. Moreover, note that $C$ is the square with vertices $0, i, 1+i, 1$, which is a simple closed contour. But $\frac{e^{z^2}}{z^2+4}$ is analytic everywhere except when $z^2 + 4 = 0$ that is when $z = \pm 2i$, which are not on or within $C$. So by Cauchy-Goursat theorem,

$$\int_C \frac{e^{z^2}}{z^2+4}\,\mathrm{d}z = 0.$$

**Exercise 5.** [6 points] Let $C$ be the following contour (its exact definition does not matter but some of its properties do):




Let $f(z) = \text{P.V.}\, z^{1/3}$ for $z \neq 0$. Evaluate the following integral (give your answer in $x + iy$ form)

$$\int_C f(z)\,\mathrm{d}z.$$

**Solution.** Let $F(z) = \frac{3}{4}\,\text{P.V.}\, z^{4/3}$ for $z \neq 0$. This function is analytic on $\mathbb{C} \setminus \mathbb{R}_-$ and we have seen that its derivative is, for any $z \in \mathbb{C} \setminus \mathbb{R}_-$,

$$F'(z) = \frac{3}{4} \cdot \frac{4}{3}\,\text{P.V.}\, z^{(4/3)-1} = \text{P.V.}\, z^{1/3} = f(z).$$

Therefore, $f$ has an antiderivative on $\mathbb{C} \setminus \mathbb{R}_-$. But the contour $C$ is included in $\mathbb{C} \setminus \mathbb{R}_-$, so by the theorem of Section 48,

$$\int_C f(z)\,\mathrm{d}z = F(-i) - F(-2+2i),$$

since $-2+2i$ is the initial point of $C$ and $-i$ the final point. With $-i = e^{-i\pi/2}$, we get

$$\begin{aligned} F(-i) &= \frac{3}{4}\exp\left(\frac{4}{3}\,\text{Log}(-i)\right) = \frac{3}{4}\exp\left(\frac{4}{3}\cdot\left(-\frac{i\pi}{2}\right)\right) = \frac{3}{4}\exp\left(-\frac{2i\pi}{3}\right) \\ &= \frac{3}{4}\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) = -\frac{3}{8} - \frac{3\sqrt{3}}{8}i. \end{aligned}$$

With $-2+2i = 2^{3/2}e^{3i\pi/4}$, we get

$$\begin{aligned} F(-2+2i) &= \frac{3}{4}\exp\left(\frac{4}{3}\,\text{Log}(-2+2i)\right) = \frac{3}{4}\exp\left(\frac{4}{3}\cdot\left(\ln(2^{3/2}) + \frac{3i\pi}{4}\right)\right) \\ &= \frac{3}{4}\exp\left(2\ln(2) + i\pi\right) = \frac{3}{4}\cdot 2^2 e^{i\pi} = -3. \end{aligned}$$

So finally we get

$$\int_C f(z)\,\mathrm{d}z = -\frac{3}{8} - \frac{3\sqrt{3}}{8}i + 3 = \frac{21}{8} - \frac{3\sqrt{3}}{8}i.$$