Probability, homework 6, due November 4.

Exercise 1. Let X be a standard Cauchy random variable. What is the density of $1/X$?

Exercise 2. Let $X_1, \dots, X_n$ be bounded, independent and identically distributed random variables such that $\mathbb{E}(X_1) = 0$, $\mathbb{E}(X_1^2) = \sigma^2$, $\mathbb{E}(X_1^4) = \kappa^4$.

(i) Calculate $\mathbb{E}\left((\sum_{k=1}^n X_i)^4\right)$.

(ii) Prove that for any $\varepsilon > 0$ and any random variable X, $\mathbb{P}(|X| > \varepsilon) \leq \varepsilon^{-4}\,\mathbb{E}(X^4)$.

(iii) Conclude that $\frac{X_1+\dots+X_n}{n}$ converges to 0 almost surely, as $n \to \infty$.

Exercise 3. Let X, Y be independent random variables with positive integers values, with distribution

$$\mathbb{P}(X = i) = \mathbb{P}(Y = i) = \frac{1}{2^i}, i \in \mathbb{N}^*.$$

Find the following proabilitities.

(i) $\mathbb{P}(\max(X, Y) \geq i)$.

(ii) $\mathbb{P}(X = Y)$.

(iii) $\mathbb{P}(X > Y)$.

Exercise 4. Let f be a continuous function on $[0, 1]$. Calculate the asymptotics, as $n \to \infty$, of

$$\int_{[0,1]^n} f\left(\frac{x_1 + \dots + x_n}{n}\right) \mathrm{d}x_1 \dots \mathrm{d}x_n.$$

Exercise 5. Let $\alpha > 0$ and, given $(\Omega, \mathcal{A}, \mathbb{P})$, let $(X_n, n \geq 1)$ be a sequence of independent real random variables with law $\mathbb{P}(X_n = 1) = \frac{1}{n^\alpha}$ and $\mathbb{P}(X_n = 0) = 1 - \frac{1}{n^\alpha}$. Prove that $\mathbb{E}(X_n) \to 0$, but that almost surely

$$\limsup_{n\to\infty} X_n = \begin{cases} 1 & \text{if} \quad \alpha \leq 1 \\ 0 & \text{if} \quad \alpha > 1 \end{cases}.$$

Exercise 6. You toss a coin repeatedly and independently. The probability to get a head is p, a tail is $1-p$. Let A_k be the following event: k or more consecutive heads occur amongst the tosses numbered $2^k, \dots, 2^{k+1} - 1$. Prove that $\mathbb{P}(A_k \text{ i.o.}) = 1$ if $p \geq 1/2$, 0 otherwise.