Probability, midterm

Grade will be scaled so that six exercises perfectly solved will give maximal score 100/100. Any types of notes, books and calculators are forbidden.

Exercise 1. Let $\varnothing \subsetneq A \subsetneq B \subsetneq \Omega$ (these are strict inclusions). What is the σ-algebra generated by $\{A, B\}$?

Exercise 2. Let $(A_n)_{n\geq 0}$ be a set of pairwise disjoint events measurable for a σ-algebra $\mathcal{A}$, and $\mathbb{P}$ a probability on $\mathcal{A}$. Show that $\lim_{n\to\infty} \mathbb{P}(A_n) = 0$.

Exercise 3. Let X be a Poisson random variable with parameter $\lambda > 0$ ($\mathbb{P}(X = k) = e^{-\lambda}\lambda^k/k!$, $k = 0, 1, \dots$). Calculate $\mathbb{E}(2^X)$.

Exercise 4. The owner of a certain website is studying the distribution of the number of visitors to the site. Every day, a million people independently decide whether to visit the site, with probability $p = 2 \times 10^{-6}$ of visiting. Calculate the probability of getting at least two visitors on a particular day. Give a good approximation.

Exercise 5. To battle against spam, Bob installs two anti-spam programs. An email arrives, which is either legitimate (event L) or spam (event L^c), and which program j marks as legitimate (event M_j) or marks as spam (event M_j^c) for $j \in \{1, 2\}$. Assume that 10% of Bobs email is legitimate and that the two programs are each "90% accurate" in the sense that $\mathbb{P}(M_j \mid L) = \mathbb{P}(M_j^c \mid L^c) = 9/10$.

(i) Find the probability that the email is legitimate, given that the 1st program marks it as legitimate.

(ii) Assume that given whether an email is spam, the two programs outputs are independent. Find the probability that the email is legitimate, given that both programs mark it as legitimate.

Exercise 6. Consider four nonstandard dice (the Efron dice), whose sides are labelled as follows (the 6 sides on each die are equally likely).

A: 4,4,4,4,0,0
B: 3,3,3,3,3,3
C: 6,6,2,2,2,2
D: 5,5,5,1,1,1

These four dice are each rolled once. Let A be the result for die A, B be the result for die B, etc.

(i) Find $\mathbb{P}(A > B)$, $\mathbb{P}(B > C)$, $\mathbb{P}(C > D)$, and $\mathbb{P}(D > A)$.

(ii) Is the event $A > B$ independent of the event $B > C$? Is the event $B > C$ independent of the event $C > D$? Explain.

Exercise 7. Let X have distribution function $F(x) = e^{-e^{-x}}$. Justify that such a probability measure on $\mathbb{R}$ exists. Let $Y = F(X)$. Calculate $\mathbb{E}(Y)$ and $\mathrm{Var}(Y)$.

Exercise 8. In a certain gambling game, a six-sided die is rolled five times; the roller wins if the last roll is the same as one of the previous rolls. What is the probability of winning this game?