Probability, midterm training.

Eight exercises of this type will be proposed, you will have 100 minutes. Grade will be scaled so that six exercises perfectly solved will give maximal score 100/100.

Exercise 1. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Prove that if $A \cap B = \emptyset$ and A, B are independent, then $\mathbb{P}(A) = 0$ or $\mathbb{P}(B) = 0$.

Exercise 2. Starting with n objects, how many different permutations are there such that none of the objects end up in their original positions?

Exercise 3. Let X be a Poisson random variable with parameter λ. Prove that for any $r \in \mathbb{N}^*$, $\mathbb{E}(X(X-1)\dots(X-r+1)) = \lambda^r$. What is the variance of X?

Exercise 4. Suppose the police have DNA evidence recovered from a crime scene. The police test a group of people and find one who matches. What is the posterior probability that the DNA is from that person, $\mathbb{P}(D \mid +)$? Suppose the sensitivity of the test is $\mathbb{P}(+ \mid D) = 0.9999$, and one minus the specificity is $\mathbb{P}(+ \mid D^c) = 0.0001$, i.e, the probability of false positive matches and false negative matches are both 0.0001. You will give the result in terms of one unknown varible, and comment.

Exercise 5. Let X be a positive random variable with density $e^{-x}\mathbb{1}_{x>0}$ (the exponential distribution). What is the density of $1/(1+X)$?

Exercise 6. Consider an ininite sequence of independent tosses of a coin that comes up heads with probability q. Let X be the number of heads required to obtain a head. Compute $\mathbb{E}(X)$.

Exercise 7. A publicity company puts grocery coupons in 10000 mailboxes. Each coupon has a chance 1/15 to be used. Let X be the number of coupons that are used. Calculate $\mathbb{E}(X)$ and $\mathrm{Var}(X)$ (give an exact answer, not an approximate one).

Exercise 8. At the lottery, you have a one in 250 millions chance to win 240 millions dollars. Let X be the amount you win. Calculate $\mathbb{E}(X)$ and $\mathrm{Var}(X)$.

Exercise 9. A standard deck of 52 cards contains 4 kings. Suppose you chose a random ordering, all 52! permutations being equally likely. Compute the following:

(i) the probability that all of the top 4 cards are kings
(ii) the probability that none of the 4 cards are kings.
(iii) the expected number of kings among the top 4 cards in the deck.

Give exact answers, not approximate ones.

Exercise 10. Suppose that X, Y, Z are independent random variables which are equal to 0 with probability 1/2, and 1 with probability 1/2. Are the events $\{X = Y\}$, $\{Y = Z\}$ and $\{X = Z\}$ pairwise independent? Are they independent?

Exercise 11. Toss 6 independent fair coins, and let X be the number that come up heads. Compute $\mathbb{E}(X)$ and $\mathrm{Var}(X)$.

Exercise 12 A coin, which you are not allowed to examine, is either a fair coin ($\mathbb{P}(\text{heads}) = 1/2$) or has two heads. Your initial opinion is $\mathbb{P}(\text{fair}) = 0.9$. The coin is flipped and heads comes up. What is your opinion now? The coin is flipped a second time and again heads comes up. What is your opinion now?

Exercise 13 Consider a routine screening test for a disease. Suppose the frequency of the disease in the population (base rate) is 0.5%. The test is highly accurate with a 5% false positive rate and a 10% false negative rate. You take the test and it comes back positive. What is the probability that you have the disease?

Exercise 14. Let $(X_n)_{n\geq 1}$ be a sequence of real random variables. Assume that $\sum_{n=1}^{\infty} \mathbb{E}(|X_n|) < \infty$. Prove that $\sum_{n=1}^{\infty} X_n$ converges almost surely and

$$\mathbb{E}\left(\sum_{n=1}^{\infty} X_n\right) = \sum_{n=1}^{\infty} \mathbb{E}\left(X_n\right).$$