Probability, homework 12, due April 29.

Exercise 1. *Kolmororov's maximal inequality and convergence of random series.* Let $(X_n)_{n\geq 1}$ be a sequence of mutually independent random variables, on the same probability space, with expectation 0 and finite variance. Let $S_n = \sum_{\ell=1}^{n} X_\ell$. Prove that for any $\lambda > 0$,

$$\lambda^2\, \mathbb{P}(\max_{1\leq k\leq n} |S_k| \geq \lambda) \leq \mathrm{Var}(S_n).$$

Prove that if $\sum_\ell \mathrm{Var}(X_\ell) < \infty$, then $(S_n)_{n\geq 1}$ converges almost surely.

Exercise 2. Let Y be an integrable random variable on $(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathcal{G}$ a sub σ-algebra of $\mathcal{A}$. Show that $|\mathbb{E}\,(Y \mid \mathcal{G})| \leq \mathbb{E}\,(|Y| \mid \mathcal{G})$.

Exercise 3. Let Y be an integrable random variable on $(\Omega, \mathcal{A}, \mathbb{P})$ and $\mathcal{G}$ a sub σ-algebra of $\mathcal{A}$. Suppose that $\mathcal{H} \subset \mathcal{G}$ is a sub σ-algebra of $\mathcal{G}$. Show that $\mathbb{E}\,(\mathbb{E}(Y \mid \mathcal{G}) \mid \mathcal{H}) = \mathbb{E}\,(Y \mid \mathcal{H})$.

Exercise 4. Let $(X_n)_{n\geq 1}$ be independent such that $\mathbb{E}(X_i) = m_i$, $\mathrm{var}(X_i) = \sigma_i^2$, $i \geq 1$. Let $S_n = \sum_{i=1}^{n} X_i$ and $\mathcal{F}_n = \sigma(X_i, 1 \leq i \leq n)$.

a) Find sequences $(b_n)_{n\geq 1}$, $(c_n)_{n\geq 1}$ of real numbers such that $(S_n^2 + b_n S_n + c_n)_{n\geq 1}$ is a $(\mathcal{F}_n)_{n\geq 1}$-martingale.

b) Assume moreover that there is a real number λ such that $e^{\lambda X_i} \in \mathrm{L}^1$ for any $i \geq 1$. Find a sequence $(a_n^{(\lambda)})_{n\geq 1}$ such that $(e^{\lambda S_n - a_n^{(\lambda)}})_{n\geq 1}$ is a $(\mathcal{F}_n)_{n\geq 1}$-martingale.

Exercise 5. Let $(X_k)_{k\geq 0}$ be i.i.d. random variables, $\mathcal{F}_m = \sigma(X_1, \ldots, X_m)$ and $Y_m = \prod_{k=1}^{m} X_k$. Under which conditions is $(Y_m)_{m\geq 1}$ a $(\mathcal{F}_m)_{m\geq 1}$-submartingale, supermartingale, martingale?

Exercise 6. Let $a > 0$ be fixed, $(X_i)_{i\geq 1}$ be iid, $\mathbb{R}^d$-valued random variables, uniformly distributed on the ball $\mathrm{B}(0, a)$. Set $S_n = x + \sum_{i=1}^{n} X_i$.

a) Let f be a superharmonic function. Show that $(f(S_n))_{n\geq 1}$ defines a supermartingale.

b) Prove that if $d \leq 2$ any nonnegative superharmonic function is constant. Does this result remain true when $d \geq 2$?