Probability, homework 5, due March 4.

Exercise 1. Build a sequence of functions $(f_n)_{n\geq 0}$, $0 \leq f_n \leq 1$, such that $\int f_n \mathrm{d}\mu \to 0$ but for any $x \in \mathbb{R}$, $(f_n(x))_{n\geq 0}$ does not converge.

Exercise 2. Let X be a random variable in $\mathrm{L}^1(\Omega, \mathcal{A}, \mathbb{P})$. Let $(A_n)_{n\geq 0}$ be a sequence of events in $\mathcal{A}$ such that $\mathbb{P}(A_N) \underset{n\to\infty}{\longrightarrow} 0$. Prove that $\mathbb{E}(X\mathbb{1}_{A_n}) \underset{n\to\infty}{\longrightarrow} 0$.

Exercise 3. Let X, Y be random variables such that X, Y ad XY are in L^1. Assume $\mathbb{E}(XY) = \mathbb{E}(X)\,\mathbb{E}(Y)$. By giving an example, prove that X and Y are not necessarily independent.

Exercise 4. Let X, Y be in L^1. Prove that, if X and Y are independent, $XY \in \mathrm{L}^1$. Show this is not true in general (i.e. if X and Y are not independent).

Exercise 5. A monkey tries to do some typesetting, by successively pushing one of the 84 keyboard buttons, forever, each button being chosen uniformly and independently of the others. Prove that almost surely, after some time, he will type the exact sequence of letters in James Joyce's Ulysses.

Exercise 6. Let $(S_n)_{n\geq 0}$ be a random walk, and denote $X_n = S_n - S_{n-1}$ for $i \geq 1$.

a) Prove that for any $A > 0$, $\mathbb{P}\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} > A\right) > 0$.

b) Read about the tail σ-algebra and Kolmogorov's o-1 law in the courses' book.

c) Prove that $\{\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} > A\} \in \cap_{n\geq 1}\sigma(X_i, i \geq n)$.

d) Deduce that $\mathbb{P}\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{n}} = +\infty\right) = 1$.