Probability, homework 9, due April 8th.

Exercise 1. Let X_λ be a real random variable, with Poisson distribution with parameter λ. Calculate the characteristic function of X_λ. Conclude that $(X_\lambda - \lambda)/\sqrt{\lambda}$ converges in distribution to a standard Gaussian, as $\lambda \to \infty$.

Exercise 2. Assume a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is such that Ω is countable and $\mathcal{A} = 2^\Omega$. Prove that convergence in probability and convergence almost sure are the same.

Exercise 3. Calculate

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!}.$$

Exercise 4. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. Bernoulli random variables, on the same probability space, with parameter $1/2$ ($\mathbb{P}(X_n = 0) = \mathbb{P}(X_n = 1) = 1/2$), and let τ_n be the hitting time of level n by the partial sums, i.e. $\tau_n = \inf\{k \mid \sum_{\ell=1}^{k} X_\ell = n\}$. Show that $n^{-1}\tau_n$ converges to 2 almost surely.

Exercise 5. Let $\alpha > 0$ and, given $(\Omega, \mathcal{A}, \mathbb{P})$, let $(X_n, n \geq 1)$ be a sequence of independent real random variables with law $\mathbb{P}(X_n = 1) = \frac{1}{n^\alpha}$ and $\mathbb{P}(X_n = 0) = 1 - \frac{1}{n^\alpha}$. Prove that $X_n \to 0$ in $\mathcal{L}^1$, but that almost surely

$$\limsup_{n\to\infty} X_n = \begin{cases} 1 & \text{if} \quad \alpha \leq 1 \\ 0 & \text{if} \quad \alpha > 1 \end{cases}.$$

Exercise 6. A sequence of random variables $(X_i)_{i\geq 1}$ is said to be completely convergent to X if for any $\varepsilon > 0$, we have $\sum_{i\geq 1} \mathbb{P}(|X_i - X| > \varepsilon) < \infty$. Prove that if the X_i's are independent then complete convergence implies almost sure convergence.

Exercise 7. Let X, Y be independent and assume that for some constant α we have $\mathbb{P}(X + Y = \alpha) = 1$. Prove that X and Y are both constant random variables.

Exercise 8. Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with mean 0 and finite variance $\mathbb{E}(X_j^2) = \sigma^2 > 0$. Let $S_n = X_1 + \cdots + X_n$. Prove that

$$\lim_{n\to\infty} \mathbb{E}\left(\frac{|S_n|}{\sqrt{n}}\right) = \sqrt{\frac{2}{\pi}}\sigma.$$

Exercise 9. Let $(X_i)_{i\geq 1}$ be a sequence of independent random variables, with X_i uniform on $[-i, i]$. Let $S_n = X_1 + \cdots + X_n$. Prove that $S_n/n^{3/2}$ converges in distribution and describe the limit.