Probability, homework 8, due November 15.

Exercise 1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $(A_n)_{n\geq 1}$ be a sequence of independent events. We denote $a_n = \mathbb{P}(A_n)$ and define $b_n = a_1 + \cdots + a_n$, $S_n = \mathbb{1}_{A_1} + \cdots + \mathbb{1}_{A_n}$. Assuming $b_n \to \infty$ as $n \to \infty$, prove that S_n/b_n converges almost surely.

Exercise 2. Let $X_1, \dots, X_n$ be i.i.d. integrable random variables, and $S = \sum_{i=1}^n X_i$. Calculate $\mathbb{E}[S \mid X_1]$ and $\mathbb{E}[X_1 \mid S]$.

Exercise 3. For fixed $a, b > 0$, let (X, Y) be a $\mathbb{N} \times \mathbb{R}_+$-valued random variable such that

$$\mathbb{P}(X = n, Y \leq t) = b \int_0^t \frac{(ay)^n}{n!} e^{-(a+b)y} \mathrm{d}y.$$

For $h : \mathbb{R}_+ \to \mathbb{R}$ continuous and bounded, calculate $\mathbb{E}[h(Y) \mid X]$. Calculate $\mathbb{E}[\frac{Y}{X+1}]$. Calculate $\mathbb{P}(X = n \mid Y)$. Calculate $\mathbb{E}[X \mid Y]$.

Exercise 4. Let (X_1, X_2) be a Gaussian vector with mean (m_1, m_2) and non-degenerate covariance matrix $(C_{ij})_{1\leq i,j\leq 2}$. Prove that

$$\mathbb{E}[X_1 \mid X_2] = m_1 + \frac{C_{12}}{C_{22}}(X_2 - m_2).$$

Exercise 5. Let X be a random variable such that $\mathbb{P}(X > t) = \exp(-t)$ for any $t \geq 0$. Let $Y = \min(X, s)$, where $s > 0$ is fixed. Prove that, almost surely,

$$\mathbb{E}[X \mid Y] = Y \mathbb{1}_{Y<s} + (1 + s)\mathbb{1}_{Y=s}.$$

Exercise 6. Let $(X_n)_{n\geq 1}$ be a sequence of nonnegative random variables on $(\Omega, \mathcal{A}, \mathbb{P})$, and $(\mathcal{F}_n)_{n\geq 0}$ a sequence of sub σ-fields of $\mathcal{F}$. Assume that $\mathbb{E}(X_n \mid \mathcal{F}_n)$ converges to 0 in probability.

(i) Show that X_n converges to 0 in probability.
(ii) Show that the reciprocal is wrong.

Exercise 7. Let μ and ν be two probability measures such that $\mu \ll \nu$ and $\nu \ll \mu$ (usually abbreviated $\mu \sim \nu$). Let $X = \frac{\mathrm{d}\mu}{\mathrm{d}\nu}$.

(i) Prove that $\nu(X = 0) = 0$.
(ii) Prove that $\frac{1}{X} = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ almost surely (for μ or ν).

Exercise 8. Let $X_1, X_2 \dots$ be i.i.d. Bernoulli random variables ($\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$) and $S_n = \sum_{i=1}^n X_i$. Which of the following sequences are Markovian? If Markovian, give the transition matrix.

(i) $(S_n^2 - n)_{n\geq 0}$.
(ii) $(S_{2n})_{n\geq 0}$.
(iii) $(|S_n|)_{n\geq 0}$.

Exercise 9. Consider a Markov chain X with state space $\{0, 1, \dots, n\}$ and transition matrix

$$\pi(0, k) = \frac{1}{2^{k+1}},\ 0 \leq k \leq n - 1,\ \pi(0, n) = \frac{1}{2^n}$$

$$\pi(k, k - 1) = 1,\ 1 \leq k \leq n - 1,\ \pi(n, n) = \pi(n, n - 1) = \frac{1}{2}.$$

(i) Prove that the chain has a unique invariant probability measure μ and calculate it.
(ii) Prove that for any $0 \leq x_0 \leq n-1$, $\pi^{(x_0+1)}(x_0, \cdot) = \mu$.
(iii) Prove that for any $0 \leq x_0 \leq n$, $\pi^{(n)}(x_0, \cdot) = \mu$.
(iv) For any $t \geq 1$, calculate

$$d(t) := \frac{1}{2} \sum_{x=0}^{n} \left| \pi^{(t)}(n, x) - \mu(x) \right|,$$

and plot $t \mapsto d(t)$.

Exercise 10. On the same probability space, let X, Y be positive random variables such that $\mathbb{E}[X \mid Y] = Y$ and $\mathbb{E}[Y \mid X] = X$ (almost surely). Prove that $X = Y$ almost surely.