Probability, practice exam.

Exercise 1.

(1) State the Borel-Cantelli lemma
(2) State the strong law of large numbers
(3) State the central limit theorem

Exercise 2. Prove that convergence in probability implies almost sure convergence along a subsequence.

Exercise 3. On the same probability space, let X and Y be two bounded random variables , i.e. there exists $C > 0$ such that $|X(\omega)| + |Y(\omega)| < C$ for any $\omega \in \Omega$. Prove that X and Y are independent if and only if for any $k, \ell \in \mathbb{N}$, we have

$$\mathbb{E}[X^k Y^\ell] = \mathbb{E}[X^k] \cdot \mathbb{E}[Y^\ell].$$

Exercise 4. Let $(p_n)_{n\geq 1}$ be a sequence of real numbers in $[0, 1]$ converging to $p \in (0, 1)$. Let Y_n be a random variable, binomial with parameters n and p_n: Y_n is equal in distribution to the sum of n independent Bernoulli random variables with parameter p_n. State and prove a central limit theorem for Y_n.

Exercise 5. Let $n \geq 2$ be fixed and consider the Markov chain corresponding to the standard random walk on $\mathbb{Z}^2 \times (\mathbb{Z}/n\mathbb{Z})$: $\pi(((x, y), z),(x', y'), z')) = \frac{1}{8}$ if $|(x, y) - (y', y')|_2 = 1$ and $z - z' = \pm 1 \mod n$, 0 otherwise. Is it transcient? Null recurrent? Positive recurrent?

Exercise 6. Let $(U_n)_{n\geq 0}$ be a sequence of i.i.d random variables, with uniform distribution on $[0, 1]$, on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(U_1, \dots, U_n)$. Define a sequence $(X_n)_{n\geq 0}$ through

$$X_0 = p \in (0, 1),\ X_{n+1} = \theta X_n + (1 - \theta)\mathbb{1}_{[0,X_n]}(U_{n+1}),$$

where $\theta \in (0, 1)$ is given.

(1) Prove that X is a $(\mathcal{F}_n)_{N\geq 0}$-martingale included in $[0, 1]$.
(2) Prove that X converges a.s. and in any L^p to a random variable denoted L.
(3) What is the distribution of L?