Probability, homework 10, due December 8.

Exercise 1. Let $(X_n)_{n\geq 1}$ be independent Gaussian such that $\mathbb{E}(X_i) = m_i$, $\mathrm{var}(X_i) = \sigma_i^2$, $i \geq 1$. Let $S_n = \sum_{i=1}^n X_i$ and $\mathcal{F}_n = \sigma(X_i, 1 \leq i \leq n)$.

a) Find sequences $(b_n)_{n\geq 1}$, $(c_n)_{n\geq 1}$ of real numbers such that $(S_n^2 + b_n S_n + c_n)_{n\geq 1}$ is a $(\mathcal{F}_n)_{n\geq 1}$-martingale.

b) Assume moreover that there is a real number λ such that $e^{\lambda X_i} \in \mathrm{L}^1$ for any $i \geq 1$. Find a sequence $(a_n^{(\lambda)})_{n\geq 1}$ such that $(e^{\lambda S_n - a_n^{(\lambda)}})_{n\geq 1}$ is a $(\mathcal{F}_n)_{n\geq 1}$-martingale.

Exercise 2. Let $(X_k)_{k\geq 0}$ be i.i.d. random variables, $\mathcal{F}_m = \sigma(X_1, \dots, X_m)$ and $Y_m = \prod_{k=1}^m X_k$. Under which conditions is $(Y_m)_{m\geq 1}$ a $(\mathcal{F}_m)_{m\geq 1}$-submartingale, supermartingale, martingale?

Exercise 3. Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration, $(X_n)_{n\geq 0}$ a sequence of integrable random variables with $\mathbb{E}(X_n \mid \mathcal{F}_{n-1}) = 0$, and assume X_n is $\mathcal{F}_n$-measurable for every n. Let $S_n = \sum_{k=0}^n X_k$. Show that $(S_n)_{n\geq 0}$ is a $(\mathcal{F}_n)_{n\geq 0}$-martingale.

Exercise 4. Let $a > 0$ be fixed, $(X_i)_{i\geq 1}$ be iid, $\mathbb{R}^d$-valued random variables, uniformly distributed on the ball $\mathrm{B}(0, a)$. Set $S_n = x + \sum_{i=1}^n X_i$.

(i) Let f be a superharmonic function. Show that $(f(S_n))_{n\geq 1}$ defines a supermartingale.

(ii) Prove that if $d \leq 2$ any nonnegative superharmonic function is constant. Does this result remain true when $d \geq 3$?

Exercise 5. Let $(S_n)_{n\geq 0}$ be a $(\mathcal{F}_n)$-martingale and τ a stopping time with finite expectation. Assume that there is a $c > 0$ such that, for all n, $\mathbb{E}(|S_{n+1} - S_n| \mid \mathcal{F}_n) < c$.

Prove that $(S_{\tau\wedge n})_{n\geq 0}$ is a uniformly integrable martingale, and that $\mathbb{E}(S_\tau) = \mathbb{E}(S_0)$.

Consider now the random walk $S_n = \sum_k^n X_k$, the X_k's being iid, $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$. For some $a \in \mathbb{N}^*$, let $\tau = \inf\{n \mid S_n = -a\}$. Prove that

$$\mathbb{E}(\tau) = \infty.$$

Exercise 6. Let X be a standard random walk in dimension 1, and for any positive integer a, $\tau_a = \inf\{n \geq 0 \mid X_{\tau_a} = a\}$. For any $\theta > 0$, calculate

$$\mathbb{E}\left((\cosh\theta)^{-\tau_a}\right).$$

Exercise 7. Let N_n be the size of a population of bacteria at time n. At each time each bacterium produces a number of offspring and dies. The number of offspring is independent for each bacterium and is distributed according to the Poisson law with rate parameter $\lambda = 2$. Assuming that $N_1 = a > 0$, find the probability that the population will eventually die, i.e. , find $\mathbb{P}(\{\text{there is } n \text{ such that } N_n = 0\})$.

Exercise 8. Let $X_n, n \geq 0$, be iid complex random variables such that $\mathbb{E}(X_1) = 0, 0 < \mathbb{E}(|X_1|^2) < \infty$. For some parameter $\alpha > 0$, let

$$S_n = \sum_{k=1}^n \frac{X_k}{k^\alpha}.$$

Prove that if $\alpha > 1/2$, S_n converges almost surely. What if $0 < \alpha \leq 1/2$?

Exercise 9. Let $(Y_n)_{n\in\mathbb{N}^*}$ be a sequence of random variables, and assume (Y_n) converges in distribution to a limiting Y. Also, on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the sequence of independent random variables $X := (X_n)_{n\in\mathbb{N}^*}$ is defined, and we assume that the sequence of partial sums $(S_n)_{n\in\mathbb{N}}$ (i.e. $S_0 = 0$ and $S_n := \sum_{j=1}^n X_j$) converges in distribution. Set $(\mathcal{F}_n)$ the natural filtration of X and $\Phi_n(t) = \mathbb{E}(\exp(\mathrm{i}tS_n))$ for $t \in \mathbb{R}$.

(i) Establish that $(\Phi_{Y_n}(\cdot))_{n\geq 1}$ converges uniformly on every compact, i.e. show that for any $a > 0$, $\max_{t\in[-a,a]} |\Phi_{Y_n}(t) - \Phi_Y(t)| \to 0$ as $n \to \infty$. Establish moreover that there exists $a > 0$ such that for any $n \geq 1$, $\min_{t\in[-a,a]} |\Phi_{Y_n}(t)| \geq 1/2$.

(ii) Show that there exists $t_0 > 0$ such that if $t \in [-t_0, t_0]$, then $(\exp(\mathrm{i}tS_n)/\Phi_n(t))_{n\geq 0}$ is a $(\mathcal{F}_n)-$martingale (i.e. both its real and imaginary parts are martingales).

(iii) Prove that we can choose $t_0 > 0$ such that for any $t \in [-t_0, t_0]$, $\lim_{n\to\infty} \exp(\mathrm{i}tS_n)$ exists $\mathbb{P}$-a.s.

(iv) Set

$$C = \{(t, \omega) \in [-t_0, t_0] \times \Omega : \lim_{n\to\infty} \exp(\mathrm{i}tS_n(\omega)) \text{ exists}\}.$$

Prove that C is measurable, i.e. in the product of $\mathcal{B}([-t_0, t_0])$ with $\mathcal{F}$.

(v) Establish that $\int_{-t_0}^{t_0} \mathbb{1}_C(t, \omega)\mathbb{P}(\mathrm{d}\omega)\mathrm{d}t = 2t_0$.

(vi) Prove that $\lim_{n\to\infty} S_n$ exists $\mathbb{P}$-a.s.