Probability, homework 5, due October 11.

As a preliminary to this homework, for exercises 3 and 4 read the Borel-Cantelli lemma (Lemma 3.4 in Varadhan's Probability Theory book).

Exercise 1. Prove that if a sequence of real random variables (X_n) converge in distribution to X, and (Y_n) converges in distribution to a constant c, then $X_n + Y_n$ converges in distribution to $X + c$.

Exercise 2. Assume that (X, Y) has joint density

$$ce^{-(1+x^2)(1+y^2)},$$

where c is properly chosen. Are X and Y Gaussian random variables? Is (X, Y) a Gaussian vector?

Exercise 3. Let $\epsilon > 0$ and X be uniformly distributed on $[0, 1]$. Prove that, almost surely (i.e. the following event has probability 1), there exists only a finite number of rationals $\frac{p}{q}$, with $p \wedge q = 1$, such that

$$\left| X - \frac{p}{q} \right| < \frac{1}{q^{2+\epsilon}}.$$

Exercise 4. You toss a coin repeatedly and independently. The probability to get a head is p, a tail is $1-p$. Let A_k be the following event: k or more consecutive heads occur amongst the tosses numbered $2^k, \dots, 2^{k+1} - 1$. Prove that $\mathbb{P}(A_k \text{ i.o.}) = 1$ if $p \geq 1/2$, 0 otherwise.

Here, i.o. stands for "infinitely often", and A_k i.o. is the event $\cap_{n\geq 1} \cup_{m\geq n} A_m$.

Exercise 5. Prove that for any $x > 0$, $\frac{1}{x} = \int e^{-tx}\mathrm{d}t$. Deduce the value of $\int_0^\infty \frac{\sin x}{x}\mathrm{d}x$.

Exercise 6. For any probability measure μ supported on $\mathbb{R}_+$, one defines the Laplace transform as

$$\mathscr{L}_\mu(\lambda) = \int_0^\infty e^{-\lambda x}\mathrm{d}\mu(x), \ \lambda \geq 0.$$

(1) Prove that $\mathscr{L}_\mu$ is well-defined, continuous on $\mathbb{R}_+$ and $\mathscr{C}^\infty$ on $\mathbb{R}_+^*$.
(2) Prove that $\mathscr{L}_\mu$ characterizes the probability measure μ supported on $\mathbb{R}_+$.
(3) Assume that for a sequence $(\mu_n)_{n\geq 1}$ of probability measure supported on $\mathbb{R}_+$, one has $\mathscr{L}_{\mu_n}(\lambda) \to \ell(\lambda)$ for any $\lambda \geq 0$, and ℓ is right-continuous at 0. Prove that $(\mu_n)_{n\geq 1}$ is tight, and that it converges weakly to a measure μ such that $\ell = \mathscr{L}_\mu$.