Probability, homework 6, due October 18.

Exercise 1. Let $(X_i)_{i\geq 1}$ be i.i.d. Gaussian with mean 1 and variance 3. Show that

$$\lim_{n\to\infty}\frac{X_1+\cdots+X_n}{X_1^2+\cdots+X_n^2}=\frac{1}{4}\text{ a.s.}$$

Exercise 2. Let f be a continuous function on $[0,1]$. Calculate the asymptotics, as $n\to\infty$, of

$$\int_{[0,1]^n} f\left(\frac{x_1+\cdots+x_n}{n}\right)\mathrm{d}x_1\dots\mathrm{d}x_n.$$

Exercise 3. The goal of this exercise is to prove that any function, continuous on an interval of $\mathbb{R}$, can be approximated by polynomials, arbitrarily close for the L^∞ norm (this is the Bernstein-Weierstrass theorem). Let f be a continuous function on $[0,1]$. The n-th Bernstein polynomial is

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right).$$

a) Let $S_n(x)=B^{(n,x)}/n$, where $B^{(n,x)}$ is a binomial random variable with parameters n and x: $B^{(n,x)}=\sum_{\ell=1}^n X_i$ where the X_i's are independent and $\mathbb{P}(X_i=1)=x$, $\mathbb{P}(X_i=0)=1-x$. Prove that $B_n(x)=\mathbb{E}(f(S_n(x)))$.

b) Prove that $\|B_n-f\|_{L^\infty([0,1])}\to 0$ as $n\to\infty$.

Exercise 4. Calculate $\lim_{n\to\infty}e^{-n}\sum_{k=0}^n\frac{n^k}{k!}$.

Exercise 5. Let $(X_n)_{n\geq 0}$ be a sequence of i.i.d random variables, with uniform distribution on $[0,1]$. Let $Y_n=(X_n)^n$.

(1) Calculate the distribution of Y_n.
(2) Show that $(Y_n)_{n\geq 0}$ converges to 0 in probability.
(3) Show that $(Y_n)_{n\geq 0}$ converges in L^1.
(4) Show that almost surely $(Y_n)_{n\geq 0}$ does not converge.

Long problem. The goal is to prove the Erdős-Kac theorem: if $w(m)$ denotes the number of distinct prime factors of m and k is a random variable uniformly distributed on $[\![1,n]\!]$, then the following convergence in distribution holds:

$$\frac{w(k)-\log\log n}{\sqrt{\log\log n}}\underset{n\to\infty}{\longrightarrow}\mathscr{N}(0,1).$$

(i) Prove that if $(X_n)_{n\geq 1}$ converges in distribution to $\mathscr{N}(0,1)$ and $\sup_{n\geq 1}\mathbb{E}[X_n^{2k}]<\infty$ for any $k\in\mathbb{N}$, then $\lim_{n\to\infty}\mathbb{E}[X_n^k]=\mathbb{E}[\mathscr{N}(0,1)^k]$ for any $k\in\mathbb{N}$.

(ii) Prove that for any $x\in\mathbb{R}$ and $d\geq 1$ we have

$$\left|e^{\mathrm{i}x}-\sum_{\ell=0}^{d}\frac{(\mathrm{i}x)^\ell}{\ell!}\right|\leq\frac{|x|^{d+1}}{(d+1)!}.$$

(iii) Assume that $\lim_{n\to\infty}\mathbb{E}[X_n^k]=\mathbb{E}[\mathscr{N}(0,1)^k]$ for any $k\in\mathbb{N}$. Prove that X_n converges in distribution to X.

(iv) Let $w_y(m)$ be the number of prime factors of m which are smaller than y. Let $(B_p)_{p\text{ prime}}$ be independent random variables such that $\mathbb{P}(B_p=1)=1-\mathbb{P}(B_p=0)=\frac{1}{p}$, $W_y=\sum_{p\leq y}B_p$, $\mu_y=\sum_{p\leq y}\frac{1}{p}$, $\sigma_y^2=\sum_{p\leq y}(\frac{1}{p}-\frac{1}{p^2})$. Prove that if $y=n^{\mathrm{o}(1)}$, then for any $d\in\mathbb{N}$ we have

$$\mathbb{E}\left[\left(\frac{w_y(k)-\mu_y}{\sigma_y}\right)^d\right]-\mathbb{E}\left[\left(\frac{W_y-\mu_y}{\sigma_y}\right)^d\right]\to 0$$

as $n\to\infty$.

(v) Conclude.