On the eigenvalues of randomized permutation matrices

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Outline of the talk

In this talk, we study some properties of the eigenvalues of random matrices, which are obtained by replacing, in random permutation matrices, the entries equal to one by more general random variables. We prove, in several cases, that the random measure associated with the point process of the eigenvalues converges, in a sense which has to be made precise, to a limit when the dimension tends to infinity. Our study is separated in two main parts: the first one corresponds to the properties which can be proved in the most general framework, and the second one focuses on the case where the non-zero entries, and then the eigenvalues, are on the unit circle.

Introduction

Random matrices have been studied for a long time, because of their relation with several problems in physics (quantum systems, statistical mechanics for example), or in statistics, and because they are related with different domains in mathematics (representation theory, number theory, etc.). The ensembles which are generally studied are classical sets of matrices like $H(N)$ (hermitian matrices of dimension N), $O(N)$ (orthogonal matrices), $U(N)$ (unitary matrices), endowed with probability measures which enjoy some remarkable properties (for example, Haar measure on $U(N)$). Generally, one studies the random set of eigenvalues of these random matrices, and in particular, their behaviour when the dimension N goes to infinity.

However, the study of classical ensembles of random matrices is difficult, because one cannot do direct computations of the quantities we wants to study. That is why less classical models can also be usefull. One of these ensembles which has been already studied in a quite detailled way is the set $\mathcal{H}_1(N)$ of permutation matrices of dimension N, i.e. matrices of linear applications of $\mathbb{C}^N$ which permute the vectors of the canonical basis. However, this matrix ensemble has some properties which can be unsatisfying if we want to compare the situation with the "classical" ensembles: for examples, all the eigenvalues are roots of unity of finite order, and one is a common eigenvalue of all the permutation matrices.

The space $G(N)$ of modified permutation matrices

The set of matrices which is considered here is the space of matrices which can be obtained from permutation matrices by replacing the entries equal to one by elements of $\mathbb{C}^*$. In dimension N, this space is a multiplicative group, denoted $\mathcal{G}(N)$. This group admits the following remarkable subgroups, the group $\mathcal{H}(N)$ of elements in $\mathcal{G}(N)$ such that their non-zero entries have modulus one, and (for $k \geq 1$) the group $\mathcal{H}_k(N)$ of elements in $\mathcal{G}(N)$ with non-zero entries lying on the set of k-th roots of unity. The spaces $\mathcal{H}(N)$ and $\mathcal{H}_k(N)$ have been already studied, for example by Wieand, but the group $\mathcal{G}(N)$ has not been studied very much.

The set of eigenvalues of a matrix in $\mathcal{G}(N)$ can be easily described in a very explicit way. More precisely, let M_N be a matrix obtained from the matrix of a permutation σ_N by replacing the non-zero entry in the row j by z_j. If the supports of the cycles of σ_N are $C_1, \dots, C_n$ and if their cardinalities are $l_1, \dots, l_n$, then the measure associated with the point process of eigenvalues of M_N is the following:

$$\mu(M_N) = \sum_{m=1}^{n} \sum_{\omega^{l_m} = Z_m} \delta_\omega,$$

where

$$Z_m = \prod_{j \in C_m} z_j.$$

This description allows us to compute explicitely some fundamental quantities related with the matrix M_N. For example, since the sum of the k-th roots of unity is zero for all $k \geq 2$, one has:

$$\mathrm{Tr}(M_N) = \sum_{j \in F} z_j,$$

where F is the set of fixed points of σ_N. More precisely, for all $k \geq 1$;

$$\mathrm{Tr}(M_N^k) = \sum_{l_m | k} l_m Z_m^{k/l_m}.$$

Let us remark that, for the moment, we have not introduced any probability measure on the space $\mathcal{G}(N)$.

Some probability measures on the space $\mathcal{G}(N)$

The probability measure on the group $\mathcal{G}(N)$ we will introduce and study depends on two elements: a strictly positive parameter θ, and a probability measure $\mathcal{L}$ on the space $\mathbb{C}^*$. It is denoted $\mathbb{P}(N, \theta, \mathcal{L})$, and it can be constructed in the following way: let σ_N be a random permutation following Ewens measure of parameter θ, and $(z_j)_{1 \leq j \leq N}$ an independent sequence of i.i.d random variables with law $\mathcal{L}$, $\mathbb{P}(N, \theta, \mathcal{L})$ is the law of the random matrix obtained from the matrix of σ_N by replacing the non-zero entry of the j-th row by z_j.

If $\mathcal{L}$ is carried by the unit circle, $\mathbb{P}(N, \theta, \mathcal{L})$ is carried by $\mathcal{H}(N)$, and if $\mathcal{L}$ is carried by the set of k-th roots of unity, $\mathbb{P}(N, \theta, \mathcal{L})$ is carried by $\mathcal{H}_k(N)$. In the sequel of this talk, we prove that the random measure associated with the point process of eigenvalues of a matrix following $\mathbb{P}(N, \theta, \mathcal{L})$ converges, in a sense which has to be made precise, to a limit random measure. In order to have a stronger convergence, in particular an almost sure convergence in distribution, it can be usefull to set all the different dimensions $N \geq 1$ into a unique probability space.

This setting is based on the so-called virtual permutations which have been introduced in the 1990's and which have been studied, for example, by Kerov, Tsilevich and Vershik. A virtual permutation is a sequence $(\sigma_N)_{N\geq 1}$ of permutations, σ_N of order N, such that σ_N is obtained by removing $N+1$ from the cycle structure of σ_{N+1} (for example, if $\sigma_5 = (125)(34)$, then $\sigma_4 = (12)(34)$). For all $\theta > 0$, there is a unique probability measure on virtual permutations such that for all $N \geq 1$, its image by the N-th coordinate is Ewens(θ) measure of order N: this probability is called Ewens measure of parameter θ on virtual permutations.

We can now introduce a probability measure $\mathbb{P}(\infty, \theta, \mathcal{L})$ on the product of the spaces $\mathcal{G}(N)$ ($N \geq 1$) in the following way: let $(\sigma_N)_{N \geq 1}$ be a virtual permutation following Ewens(θ) measure, and $(z_j)_{j \geq 1}$ an independent sequence of i.i.d. random variables with law $\mathcal{L}$; $\mathcal{P}(\infty, \theta, \mathcal{L})$ is the distribution of $(M_N)_{N \geq 1}$ ($M_N \in \mathcal{G}(N)$), where M_N is obtained from the matrix of σ_N by replacing the non-zero entry in its j-th row by z_j. One easily see that if $(M_N)_{N \geq 1}$ follows the distribution $\mathbb{P}(\infty, \theta, \mathcal{L})$, then M_N follows $\mathbb{P}(N, \theta, \mathcal{L})$ for all $N \geq 1$.

The "non-unitary case"

In this part of the talk, $\mathcal{L}$ is not supposed to be carried by the unit circle. Therefore, an eigenvalue of a matrix M_N following $\mathbb{P}(N, \theta, \mathcal{L})$ can, a priori, be any element of $\mathbb{C}^*$. A natural question one can ask here is the asymptotics of the "empirical measure" of the eigenvalues, i.e. the random probability measure $\mu(M_N)/N$. Since this measure is dominated by the large cycles of the permutation σ_N associated to M_N and since a cycle of length l gives the l-th roots of the product of l i.i.d. random variables of law $\mathcal{L}$, one can expect, by law of large numbers, that under some quite general assumptions on $\mathcal{L}$, $\mu(M_N)/N$ is close to the uniform measure on a circle centered at the origin.

More precisely, let $(M_N)_{N \geq 1}$ be a sequence of matrices following the law $\mathbb{P}(\infty, \theta, \mathcal{L})$. We suppose that if Z is a random variable with law $\mathcal{L}$, then $\log(|Z|)$ is in L^1, and we set $R := \exp(\mathbb{E}[\log(|Z|)])$. Then, almost surely, the empirical measure $\mu(M_N)/N$ converges weakly to the uniform measure on the circle of center zero and radius R. We remark that this coupling of all the dimensions $N \geq 1$ is used here. If we suppose only that M_N follows the law $\mathbb{P}(N, \theta, \mathcal{L})$ for each N, then we deduce that for all bounded, continuous functions f from $\mathbb{C}$ to $\mathbb{R}$, the integral of f with respect to $\mu(M_N)$ converges in probability to the average of f on the circle of radius R.

Since, for a large class of measures $\mathcal{L}$, a large part of the eigenvalues are close to a fixed circle, it is natural to study the law of the eigenvalues which are far from this circle (say, at a distance greater than or equal to ε, for a given $\varepsilon > 0$). We have also a result related to this problem. For all $N \geq 1$, let M_N be a matrix which follows the distribution $\mathbb{P}(N, \theta, \mathcal{L})$. We suppose that if Z has law $\mathcal{L}$, then $\log(|Z|)$ is in L^4, and we set, as above, $R := \exp(\mathbb{E}[\log(|Z|)])$. Then, there exists a random measure μ_∞ such that for any continuous, bounded function f, equal to zero in a neighborhood of the circle $\{|z| = R\}$, the integral of f with respect to $\mu(M_N)$ tends in distribution to the integral of f with respect to μ_∞, which is a.s. finite.

We remark that here, the measure $\mu(M_N)$ is not renormalized. Moreover, the convergence stated here cannot be replaced by an almost sure convergence if we suppose that $(M_N)_{N\geq 1}$ follows the distribution $\mathbb{P}(\infty, \theta, \mathcal{L})$: this setting is irrelevant here. Moreover, the distribution of μ_∞ can be explicitly described, one has:

$$\mu_\infty = \sum_{k=1}^{\infty} \sum_{p=1}^{a_k} \sum_{\omega^k = T_{k,p}} \delta_\omega,$$

where all the variables $(a_k)_{k\geq 1}$ and $(T_{k,p})_{p\geq 0}$ are independent, a_k is a Poisson random variable of parameter θ/k, and $T_{k,p}$ follows the multiplicative convolution of k copies of $\mathcal{L}$. This measure μ_∞ has a.s. an infinite total mass, and its description is related to the structure of the small cycles of random permutations following Ewens(θ) measure (the cycle structure of random permutations has been studied in a very detailled way, in particular by Aratia, Barbour and Tavaré).

Another problem is the computation of the average $\tilde{\mu}_N$ of $\mu(M_N)$, or the average $\tilde{\mu}_\infty$ of μ_∞. These (deterministic) measures can be described as follows: for all Borel sets $A \subset \mathbb{C}$, the expectation of $\mu(M_N)(A)$ is $\tilde{\mu}_N(A)$ and the expectation of $\mu_\infty(A)$ is $\tilde{\mu}_\infty(A)$. One has explicit expressions: for M_N following the distribution $\mathbb{P}(N, \theta, \mathcal{L})$,

$$\tilde{\mu}_N = \theta \sum_{k=1}^{N} \frac{N(N-1)\dots(N-k+1)}{(N-1+\theta)\dots(N-k+\theta)} \mathcal{L}_k \,,$$

$$\tilde{\mu}_\infty = \theta \sum_{k=1}^{N} \mathcal{L}_k \,,$$

where $\mathcal{L}_k$ is the unique probability, invariant by multiplication by k-th roots of unity, and such that its image by the k-th power is the multiplicative convolution of k copies of $\mathcal{L}$.

Moreover, if for Z following the law $\mathcal{L}$, $\log(|Z|)$ is integrable, then $\tilde{\mu}_N$ converges weakly to the uniform measure on the circle $\{z = R\}$ (R defined as before). If $\log(|Z|)$ is in L^4, then for $0 < R_1 < R < R_2$, the restriction of $\tilde{\mu}_N$ to the set $\{|z| \notin (R_1, R_2)\}$ converges weakly to the restriction of $\tilde{\mu}_\infty$ to $\{|z| \notin (R_1, R_2)\}$.

The "unitary case"

In this part, we suppose that $\mathcal{L}$ is carried by the unit circle, therefore $\mathbb{P}(N,\theta,\mathcal{L})$ is carried by $\mathcal{H}(N)$ for all $N \geq 1$. In this case, the corresponding eigenvalues are on the unit circle, and then, we can replace them by eigenangles. More precisely, since the average spacing of eigenangles is $2\pi/N$, we will study the eigenangles, scaled by a factor $N/2\pi$, in order to have a fixed average spacing (equal to 1). Hence, if M_N is a matrix in $\mathcal{H}(N)$, let us define the measure:

$$\tau_N(M_N) := \sum_{x\in\mathbb{R}, e^{2\pi x/N}\in E(M_N)} m(e^{2\pi x/N})\delta_x,$$

where E is the set of eigenvalues of M_N and $m(z)$ is the multiplicity of z as an eigenvalue of M_N.

Let us now suppose that $(M_N)_{N \geq 1}$ follows the distribution $\mathbb{P}(\infty, \theta, \mathcal{L})$. If we denote by $(\sigma_N)_{N \geq 1}$ the virtual permutation associated with $(M_N)_{N \geq 1}$, there exists a partition $(C_m)_{m \geq 1}$ of the set of strictly positive integers (the sets C_m are ordered by increasing smallest elements), such that for $N \geq 1$, the supports of the cycles of σ_N are the intersections $C_{N,m}$ of C_m and $\{1, \dots, N\}$. We denote by $l_{N,m}$ the cardinality of $C_{N,m}$, we set $y_{N,m} := l_{N,m}/N$, and we define $\gamma_{N,m} \in \mathbb{R} \backslash \mathbb{Z}$ by:

$$\gamma_{N,m} = \frac{1}{2i\pi} \sum_{j \in C_{N,m}} \log(z_j),$$

where z_j is the non-zero entry of the j-th row of M_N, for $N \geq j$. With this notation, one has:

$$\tau_N(M_N) = \sum_{m=1}^{\infty} \mathbb{1}_{y_{N,m} > 0} \sum_{k \in \mathbb{Z}} \delta_{(\gamma_{N,m} + k)/y_{N,m}}.$$

Now, by Tsilevich, the sequence $y_{N,m}$ converges a.s. to a limit y_m and $(y_m)_{m\geq 1}$ follows a GEM process of parameter θ. If $\mathcal{L}$ is Dirac measure at 1, one obtain a limit random measure

$$\tau_\infty((M_N)_{N\geq 1}) = \sum_{m=1}^{\infty} \sum_{k\in\mathbb{Z}} \delta_{k/y_m}$$

in the following sense: for all positive, continuous functions f with compact support, the integral of f with respect to $\tau_N(M_N)$ tends a.s. to the integral of f with respect to $\tau_\infty((M_N)_{N\geq 1})$. Note that $\tau_\infty((M_N)_{N\geq 1})$ has an infinite mass at zero, that is why we need positive functions f.

For more general $\mathcal{L}$, one cannot expect almost sure convergence because $\gamma_{N,m}$ does not converge when N goes to infinity. However, let us suppose that $\mathcal{L}$ is uniform measure on the unit circle (and then set $k = \infty$), or uniform measure on the set of k-th roots of unity, for some integer $k \geq 1$. Then the distribution of $\tau_N(M_N)$ converges to the distribution of

$$\tau_\infty := \sum_{m=1}^{\infty} \sum_{k \in \mathbb{Z}} \delta_{(k+\gamma_m)/x_m},$$

where $(\gamma_m)_{m\geq 1}$ is an i.i.d. sequence of random variables with uniform distribution on the k-th roots of unity for $k < \infty$, or uniform distribution on the unit circle for $k = \infty$, and $(x_m)_{m\geq 1}$ is an independent GEM (or Poisson-Dirichlet) process of parameter θ.

The convergence stated above has to be understood in the following way: let f be a continuous, positive function with compact support.

- If $f(0) > 0$ and $k < \infty$, then the integral of f with respect to τ_∞ is a.s. infinite, and for all $A > 0$, the probability that the integral of f with respect to $\tau_N(M_N)$ is smaller than A tends to zero when N goes to infinity.
- If $f(0) = 0$ or $k = \infty$, then the integral of f with respect to $\tau_N(M_N)$ converges in distribution to then the integral of f with respect to τ_∞, which is a.s. finite.

Questions

One can now ask several questions, in this unitary case:

- Do we have the convergence in distribution stated above for all the distributions $\mathcal{L}$ carried on the unit circle?
- Can we compute the correlation functions corresponding to the point process of eigenangles or its scaling limit?
- Can we compute the law of the first positive eigenangle, or the first positive point of the limit point process?
- Can we precise the rate of convergence of the empirical measure of eigenvalues towards the uniform measure on the unit circle? This problem has been partially studied by Wieand, and also, in the case of Haar measure on $U(N)$, by Diaconis and Evans, for example.
- Can we do a similar coupling of all the dimensions for general unitary matrices, and can we deduce almost sure convergence in this more classical framework?