Stochastic analysis, homework 3.

In all the following, B is a Brownian motion, with completed right-continuous filtration $\mathcal{F}_t = \sigma(B_s, s \leq t)$.

Exercise 1 Let c and d be two strictly positive numbers, B a standard Brownian motion and $T = T_c \wedge T_{-d}$.

1 Prove that, for every real number s,

$$\mathbb{E}\left(e^{-\frac{s^2}{2}T}\mathbb{1}_{T=T_c}\right) = \frac{\sinh(sd)}{\sinh(s(c+d))}.$$

Prove that

$$\mathbb{E}\left(e^{-\frac{s^2}{2}T}\right) = \frac{\cosh(s(c-d)/2)}{\cosh(s(c+d)/2)}.$$

2 Prove that for $0 \leq s < \pi/(c+d)$,

$$\mathbb{E}\left(e^{\frac{s^2}{2}T}\right) = \frac{\cos(s(c-d)/2)}{\cos(s(c+d)/2)}.$$

For this, use either a clearly justified analytic continuation or the complex martingale $e^{\mathrm{i}s(B_t-(c-d)/2)+s^2t/2}$.

Exercise 2 Let f be a continuous function on $\mathbb{R}$.

1 Prove that if $(f(B_t), t \geq 0)$ is a $(\mathcal{F}_t)_{t\geq 0}$ martingale, then f is affine.

2 Suppose that $(f(B_t), t \geq 0)$ is a $(\mathcal{F}_t)_{t\geq 0}$ submartingale: prove that f has no proper local maximum. Hint: for $c > 0$, use the stopping times $T = T_c \wedge T_{-1}$ and $S = \inf\{t \geq T : B_t = -1 \text{ } or \text{ } c+\epsilon \text{ } or \text{ } c-\epsilon\}$.

3 Suppose that $(f(B_t), t \geq 0)$ is a $(\mathcal{F}_t)_{t\geq 0}$ submartingale: prove that f is convex.

Exercise 3 For $a > 0$, let $\sigma_a = \inf\{t \geq 0 : B_t < t - a\}$.

1 Prove that σ_a is a.s. finite and that $\lim_{a\to\infty} \sigma_a = \infty$ almost surely.

2 Prove that $\mathbb{E}(e^{\frac{1}{2}\sigma_a}) = e^a$. For this, use the martingale $e^{-(\sqrt{1+2\lambda}-1)(B_t-t)-\lambda t}$, and an analytic continuation.

3 Prove that the martingale $e^{B_t-\frac{1}{2}t}$ stopped at σ_a is uniformly integrable.

4 For $a > 0$ and $b > 0$, define $\sigma_{a,b} = \inf\{t \geq 0 : B_t < bt - a\}$. Prove that $\sigma_{a,b} \overset{\text{law}}{=} b^{-2}\sigma_{ab,1}$ and

$$\mathbb{E}(e^{\frac{1}{2}b^2\sigma_{a,b}}) = e^{ab}.$$

5 For $b < 1$, prove that $\mathbb{E}(e^{\frac{1}{2}\sigma_{1,b}}) = \infty$.