Stochastic analysis, homework 4.

Exercise 1 Let B be a standard Brownian motion and $M_t = \int_0^t \mathbb{1}_{B_s=0} \mathrm{d}B_s$. Prove that M is indistinguishable from 0.

Exercise 2 Let $X_t = \int_0^t (\sin s)\mathrm{d}B_s$. Prove that this is a Gaussian process. What are $\mathbb{E}(X_t)$ and $\mathbb{E}(X_s X_t)$? Prove that

$$X_t = (\sin t)B_t - \int_0^t (\cos s)B_s \mathrm{d}s.$$

Exercise 3 Let B be a Brownian motion. For $n \geq 0$, let

$$H_n(x,y) = (\partial_\alpha)^n \mid_{\alpha=0} e^{\alpha x - \frac{\alpha^2}{2}y}.$$

Prove that for $n = 1, 2, 3$, $H_n(B_t, t)$ is a martingale. Prove this for any n.

Exercise 4 Prove that if f is a deterministic continuous square integrable function,

$$\mathbb{E}\left(B_t \int_0^\infty f(s)\mathrm{d}B_s\right) = \int_0^t f(s)\mathrm{d}s.$$

Exercise 5 If M is a continuous local martingale with $M_0 = 0$, prove that almost surely

$$\{(e^{M_t - \frac{1}{2}\langle M\rangle_t})_{t=\infty} = 0\} = \{\langle M\rangle_\infty = \infty\},$$

i.e. the symmetric difference between both events has measure 0.

Exercise 6

1 Let B be a Brownian motion and H a continuous adapted process with respect to the natural filtration of B. Assume that almost surely, for any t, $\int_0^t H_s^2 \mathrm{d}s < \infty$ and $\int_0^\infty H_s^2 \mathrm{d}s = \infty$. Set

$$T = \inf\left\{t \geq 0 \mid \int_0^t H_s^2 \mathrm{d}s = \sigma^2\right\}.$$

Prove that $\int_0^T H_s \mathrm{d}B_s \sim \mathcal{N}(0, \sigma^2)$.

2 Let $(B^n, n \geq 1)$ be standard Brownian motions defined on the same probability space, and for any n let K^n be continuous, adapted with respect to the filtration of B^n, such that for some constants T_n

$$\int_0^{T_n} (K_s^n)^2 \mathrm{d}s \underset{n\to\infty}{\longrightarrow} \sigma^2.$$

Prove that $\int_0^{T_n} K_s^n \mathrm{d}B_s^n$ converges in law to $\mathcal{N}(0, \sigma^2)$.

Exercise 7 Let B^1 and B^2 be independent Brownian motions, defined on the same probability space. Let

$$X_t = e^{B_t^1} \int_0^t e^{-B_s^1} \mathrm{d}B_s^2, \; Z_t = \sinh B_t^1.$$

Prove that both processes have the same law.