Stochastic calculus, final exam

Lecture notes are not allowed. Six exercises perfectly solved give the maximum grade 100/100.

Exercise 1. Let $X_n, n \geq 0$, be independent random variables. Assume that $\mathbb{E}(X_j) = 0$ and there exists a $\beta > 0$ such that $\mathbb{E}(|X_j|^2) = j^{-\beta}$ for any $j \geq 1$. Let $S_n = \sum_{k=1}^n X_k$. For which values of β does S converge almost surely? Prove it.

Exercise 2. Let B be a Brownian motion starting at $x > 0$, and $T_0 = \inf\{s \geq 0 : B_s = 0\}$. What is the distribution of $\sup_{t \leq T_0} B_t$? You can apply a stopping time theorem to a positive martingale.

Exercise 3. Let B be a standard Brownian motion and $M_t = \max_{0 \leq s \leq t} B_s$.

(1) Explain why M_t has the same distribution as $\sqrt{t} M_1$.
(2) What is the density of M_t?

Exercise 4. Let $(S_n)_{n \geq 0}$ be a standard random walk.

(i) State Donsker's theorem for $(S_n)_{n \geq 0}$.
(ii) As $N \to \infty$, find the asymptotics for

$$\mathbb{E}\left(\max_{N/2 < n < N} |S_n|\right).$$

Exercise 5.

(i) Write the multidimensional Itô formula.
(ii) Let B^1 and B^2 be independent Brownian motions, defined on the same probability space. Let

$$X_t = e^{B_t^1} \int_0^t e^{-B_s^1} \mathrm{d}B_s^2.$$

What simple stochastic differential equation does it satisfy?

Exercise 6. Sketch the proof that the Brownian motion is transcient in dimension $d \geq 3$.

Exercise 7. Let $a > 0$, $\gamma \geq 0$, and $T_{a,\gamma} = \inf\{t \geq 0 \mid B_t + \gamma t = a\}$. Prove that the density of $T_{a,\gamma}$ with respect to the Lebesgue measure on $\mathbb{R}_+$ is

$$\frac{a}{\sqrt{2\pi t^3}} e^{\frac{-(a-\gamma t)^2}{2t}}.$$

If you solve the case $\gamma = 0$, you get 3/4 of the points.

Exercise 8. Assume the process $(X_t)_{t \geq 0}$ satisfies $\mathrm{d}X_t = X_t(\mu_t \mathrm{d}t + \sigma_t \mathrm{d}B_t)$ for some Brownian motion B which corresponds to the Wiener measure $\mathbb{P}$.

(i) Prove that $X_t e^{-\int_0^t \mu_s \mathrm{d}s}$ is a local martingale under $\mathbb{P}$.
(ii) Find a probability $\mathbb{Q}$ under which X is a local martingale.
(iii) Find a probability $\widetilde{\mathbb{Q}}$ under which X^{-1} is a local martingale.

Exercise 9. Let B be a standard Brownian motion. Justify the following stochastic differential equation has only one solution. In which sense? What does that mean?

$$\mathrm{d}X_t = X_t \mathrm{d}t + (1 - e^{-|X_t|}) X_t \mathrm{d}B_t.$$

Find the solution.

Exercise 10.

(i) State the Feynman-Kac theorem.
(ii) Let B be a Brownian motion starting at 0. What partial differential equation does $\mathbb{E}\left(e^{\int_t^T B_s^2 \mathrm{d}s}\right)$ satisfy? What are the boundary conditions?