Stochastic calculus, homework 10, due December 5th.

Below B is a standard Brownian motion, adapted with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$.

Exercise 1. We consider the stochastic differential equation $\mathrm{d}X_t = \alpha X_t \mathrm{d}t + \beta X_t \mathrm{d}B_t$, $X_0 = 0$.

(i) Prove that $X_t = e^{(\alpha - \frac{\beta^2}{2})t + \beta B_t}$ is a solution.

(ii) Show that, for $\alpha \geq 0$, X is a submartingale with respect to $(\mathcal{F}_t)_{t\geq 0}$. For which α is it a martingale?

Exercise 2. We consider the stochastic differential equation $\mathrm{d}X_t = (b + \beta X_t)\mathrm{d}B_t$, $X_0 = x$ with $x \neq -b/\beta$.

(i) For any $y \neq -b/\beta$, we define $h(y) = \frac{1}{\beta} \log \left| \frac{b+\beta y}{b+\beta x} \right|$. What equation does $Y_t = h(X_t)$ satisfy?

(ii) What is the solution to the initial stochastic differential equation?

Exercise 3. We consider the stochastic differential equation $\mathrm{d}X_t = a(b - X_t)\mathrm{d}t + \sigma\sqrt{X_t}\mathrm{d}B_t$, $X_0 = x$ with $x > 0$. Assume there exists a solution in the strongest sense you want, and that this solution is as integrable as you want.

Calculate the expectation and variance of X_t

Exercise 4. We consider the stochastic differential equation $\mathrm{d}X_t = -\alpha^2 X_t^2 (1 - X_t)\mathrm{d}t + \alpha X_t(1 - X_t)\mathrm{d}B_t$, $X_0 = x$ with $x \in (0, 1)$.

(i) Write a program to simulate a trajectory and show a sample plot.

(ii) Let $Y_t = X_t/(1 - X_t)$. What stochastic differential equation does Y satisfy?

(iii) Show that

$$X_t = \frac{x e^{\alpha B_t - \alpha^2 \frac{t}{2}}}{x e^{\alpha B_t - \alpha^2 \frac{t}{2}} + 1 - x}$$

is a solution.