Stochastic calculus, homework 5, due October 24th.

Exercise 1. Amongst the following processes, which ones are $\mathcal{F}$-martingales, where $\mathcal{F}$ is the natural filtration of $(B_s, s \geq 0)$ (i.e. $\mathcal{F}_t = \sigma(B_u, 0 \leq u \leq s)$)? $B_t^2 - t$, $B_t^3 - 3\int_0^t B_s \mathrm{d}s$, $B_t^3 - 3tB_t$, $tB_t - \int_0^t B_s \mathrm{d}s$.

Exercise 2. Let B and $\widetilde{B}$ be two independent standard Brownian motions, and $\rho \in [0,1]$. Prove that $\rho B + \sqrt{1-\rho^2}\widetilde{B}$ is a standard Brownian motion.

Let $B = (B^{(1)}, \dots, B^{(d)})^t$ be a column vector with independent standard Brownian motions as entries. Let U be an orthogonal matrix. Show that the entries of UB are also independent Brownian motions.

Exercise 3. Prove that

$$\mathbb{E}\left((S_t - K)_+\right) = x\mathcal{N}(d_1(t)) - K\mathcal{N}(d_2(t))$$

where $S_t = xe^{\sigma B_t - \frac{\sigma^2}{2}t}$ and B is a standard Brownian motion. In the above formula, we used the notations $d_1(t) = \frac{1}{\sigma\sqrt{t}} \log\left(\frac{x}{K} + \frac{\sigma^2 t}{2}\right)$, $d_2(t) = d_1(t) - \sigma\sqrt{t}$, and $\mathcal{N}(x) = \int_{-\infty}^x \frac{e^{-u^2/2}}{\sqrt{2\pi}} \mathrm{d}u$.

Exercise 4. Let c and d be two strictly positive numbers, B a standard Brownian motion and $T = T_c \wedge T_{-d}$.

Prove the following Laplace transform identity: for every real number s,

$$\mathbb{E}\left(e^{-\frac{s^2}{2}T}\right) = \frac{\cosh(s(c-d)/2)}{\cosh(s(c+d)/2)}.$$

By Taylor-expanding in s, what are the expectation and variance of T?

Hint for the Laplace transform: follow the usual strategy applying a stopping time theorem to a pertinent martingale. This martingale is of exponential type.

Exercise 5: *bonus.* Prove that

$$\mathbb{P}(\sup_{s \leq u \leq t} B_u > 0, B_s < 0) = 2\mathbb{P}(B_t > 0, B_s < 0) = 2\left(\frac{1}{4} - \frac{1}{2\pi} \arcsin\sqrt{\frac{s}{t}}\right).$$

What is the distribution of $g_t = \sup\{s \leq t : B_s = 0\}$?