Stochastic calculus, midterm exam

Lecture notes are not allowed. Six exercises perfectly solved give the maximum grade 100/100.

Exercise 1. Let $(X_i)_{i\geq 1}$ be i.i.d. uniform on $[0,1]$. What is the limit of

$$\frac{X_1+\cdots+X_n}{X_1^2+\cdots+X_n^2}$$

as $n\to\infty$?

Exercise 2. Let Y_1, Y_2, Y_3 be independent $\mathscr{N}(0,1)$ random variables. Let

$$X_1 = Y_2+2Y_3,\ X_2 = Y_1-3Y_2+xY_3.$$

for some real number x.

(i) Explain why $\mathbf{X}=(X_1,X_2)$ is a Gaussian vector.
(ii) For what values of x are X_1 and X_2 independent?
(iii) Give an example of a vector with Gaussian entries which is not a Gaussian vector.

Exercise 3. Let X be a real Gaussian random variable with mean 0 and variance 7. Calculate

(i) $\mathbb{E}(e^{\lambda X})$ for any $\lambda\in\mathbb{C}$;
(ii) $\mathbb{E}(X^7-3X^2+12X-4)$.

Exercise 4. Let $X_1, X_2, \dots$ be independent random variables and $\mathbb{P}(X_j=1)=\mathbb{P}(X_j=-1)=\mathbb{P}(X_j=0)=1/3$ for any j. Let $S_n = X_1+\cdots+X_n$ and $\mathcal{F}_n=\sigma(X_1,\dots,X_n)$.

(i) Calculate $\mathbb{E}(S_n)$, $\mathbb{E}(S_n^2)$.
(ii) Is S_n a $(\mathcal{F}_n)_{n\geq 0}$-martingale?
(iii) If $m<n$, calculate $\mathbb{E}(S_n^2\mid\mathcal{F}_m)$.
(iv) If $m<n$, calculate $\mathbb{E}(X_m\mid S_n)$.

Exercise 5. Let $X_i, i\geq 1$, be i.i.d. random variables, $X_i\geq 0, \mathbb{E}(X_i)=1$. Prove that if $Y_n=\prod_1^n X_k$, $\mathcal{F}_n=\sigma(X_k, k\leq n)$, $(Y_n)_{n\geq 0}$ is a $(\mathcal{F}_n)$-martingale.

Prove that if $\mathbb{P}(X_1=1)<1$, Y_n converges to 0 almost surely.

Exercise 6. Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. random variables with mean 0 and finite variance $\mathbb{E}(X_j^2)=\sigma^2>0$. Let $S_n=X_1+\cdots+X_n$.

(i) State Donsker's theorem.
(ii) Prove that $\lim_{n\to\infty}\mathbb{E}\left(\frac{|S_n|}{\sqrt{n}}\right)=\sqrt{\frac{2}{\pi}}\sigma$.

Exercise 7. Let B be a Brownian motion. Calculate $\mathbb{E}\left(e^{\int_0^1 B_s\mathrm{d}s}\right)$.

Exercise 8. Let B be a Brownian motion and $\mathcal{F}_t=\sigma(B_s, s\leq t)$.

(i) Show that $(B_t^2-t)_{t\geq 0}$ is a $(\mathcal{F}_t)_{t\geq 0}$-martingale.
(ii) Is there a deterministic function $(f_t)_{t\geq 0}$ such that $(e^{(B_t^2-t)-f_t})_{t\geq 0}$ is a $(\mathcal{F}_t)_{t\geq 0}$-martingale?

Exercise 9. State the stopping time theorem for uniformly integrable continuous martingales. What are the main intermediate lemmas for the proof?

Exercise 10. Let $X_n, n\geq 0$, be independent random variables. Assume that $\mathbb{E}(X_j)=0$ and there exists a $\alpha>0$ such that $\mathbb{E}(|X_j|^2)=j^{-\alpha}$ for any $j\geq 1$. Let $S_n=\sum_{k=1}^n X_k$. For which values of α does S converge almost surely?