Stochastic calculus, practice for the final exam

These are just examples of typical exercises for the final exam. They need to be supplemented by all exercises from the midterm practice. You may also be asked to state important theorems and give some short proofs of results seen in class. Lecture notes will not be allowed.

In the final, out of ten exercises, six will be identical or very similar to exercise from the following list.

(1) All exercises in all homeworks, except the bonus ones.
(2) All exercises in the midterm practice exam.
(3) The following four additional exercises.

Exercise 1. Give an example of convergence almost sure with no convergence in L^1, both in discrete and continuous time. Prove it.

Exercise 2. Give an example of sum of independent variables for which the central limit theorem does not hold. Prove it. Same question with the law of large numbers.

Exercise 3. Calculate $\mathbb{E}\left(B_s^2 e^{\mu B_s - \frac{\sigma^2}{2}s}\right)$.

Exercise 4. Let $a>0$, $\gamma \geq 0$, and $T_{a,\gamma} = \inf\{t \geq 0 \mid B_t + \gamma t = a\}$. Prove that the density of $T_{a,\gamma}$ with respect to the Lebesgue measure on $\mathbb{R}_+$ is

$$\frac{a}{\sqrt{2\pi t^3}} e^{\frac{-(a-\gamma t)^2}{2t}}.$$

Exercise 5. Consider the stochastic differential equation $\mathrm{d}X_t = \alpha X_t \mathrm{d}t + \sigma X_t \mathrm{d}B_t$, with $X_0 \in [a,b]$. Calculate $\mathbb{E}(T)$ where $T = \inf\{s \geq 0 \mid X_s \notin [a,b]\}$.

Exercise 6. Let B be a Brownian motion starting at x. Using an argument based on partial differential equations, calculate $\mathbb{E}\left(\exp(\int_t^T B_s \mathrm{d}s)\right)$