Stochastic calculus, homework 1, due September 18.

Exercise 1. State the central limit theorem for partial sums from a sequence of i.i.d. Bernoulli random variables $(X_i)_{i\geq 1}$, where $\mathbb{P}(X_i = 1) = p$, $\mathbb{P}(X_i = -1) = 1 - p$, $p \in [0, 1]$.

Exercise 2. Let $(X_i)_{i\geq 1}$ be i.i.d. Gaussian with mean 1 and variance 3. Show that

$$\lim_{n\to\infty} \frac{X_1 + \cdots + X_n}{X_1^2 + \cdots + X_n^2} = \frac{1}{4} \text{ a.s.}$$

Exercise 3. Find an example of real random variables $(X_n)_{n\geq 1}$, X, in L^1, such that $(X_n)_{n\geq 1}$ converges to X in distribution and $\mathbb{E}(X_n)$ converges, but not towards $\mathbb{E}(X)$.

Exercise 4. Let $(X_n)_{n\geq 1}$ be independent Gaussian such that $\mathbb{E}(X_i) = m_i$, $\mathrm{var}(X_i) = \sigma_i^2$, $i \geq 1$. Let $S_n = \sum_{i=1}^n X_i$ and $\mathcal{F}_n = \sigma(X_i, 1 \leq i \leq n)$.

a) Find sequences $(b_n)_{n\geq 1}$, $(c_n)_{n\geq 1}$ of real numbers such that $(S_n^2 + b_n S_n + c_n)_{n\geq 1}$ is a $(\mathcal{F}_n)_{n\geq 1}$-martingale.

b) Let $\lambda \in \mathbb{R}$. Find a sequence $(a_n^{(\lambda)})_{n\geq 1}$ such that $(e^{\lambda S_n - a_n^{(\lambda)}})_{n\geq 1}$ is a $(\mathcal{F}_n)_{n\geq 1}$-martingale.

Exercise 5. The goal of this exercise is to justify simulation of Gaussian random variables from uniform ones.

Let U_1 and U_2 be two independent random variables, uniform on $[0, 1]$, $\theta = 2\pi U_1$ and $S = -\log U_2$.

(i) Prove that S has an exponential distribution.

(ii) Prove that $R = \sqrt{2S}$ has density $xe^{-x^2/2}$ on $\mathbb{R}_+$. This is the Rayleigh distribution.

(iii) Prove that $X_1 = R\cos\theta$ and $X_2 = R\sin\theta$ are independent Gaussian random variables. This is the Box-Muller method to simulate Gaussian random variables.