Stochastic calculus, homework 2, due September 25.

Exercise 1. Let $(X_n, n \geq 0)$ be a non-negative supermartingale. Show the following maximal ineqality: for $a > 0$,

$$a\mathbb{P}\left(\sup_{[\![0,n]\!]} X_k > a\right) \leq \mathbb{E}(X_0).$$

Exercise 2. Let $X_0 > 0$, and at time $n+1$ you get $\epsilon_n Y_n$ where Y_n was your stake at time n, the ϵ_n's are iid and $\mathbb{P}(\epsilon_n = 1) = p = 1 - \mathbb{P}(\epsilon_n = -1)$, $p \in (1/2, 1)$: what you own at time $n+1$ is

$$X_{n+1} = X_n + \epsilon_{n+1} Y_n,$$

where $Y_n \in \mathcal{F}_n$, $0 \leq Y_n \leq X_n$, $\mathcal{F}_n = \sigma(\epsilon_1, \dots, \epsilon_n)$.

You want to maximize the expected return $\mathbb{E}\left(\log \frac{X_n}{X_0}\right)$, by finding the good strategy, i.e. what suitable $\mathcal{F}_n$-measurable function Y_n to choose. Prove that for some $\lambda \geq 0$ explicit in terms of p, $((\log X_n) - n\lambda, n \geq 0)$ is a $(\mathcal{F}_n)$-supermartingale, so that

$$\mathbb{E}\left(\log \frac{X_n}{X_0}\right) \leq n\lambda.$$

Find a strategy such that equality occurs in the above equation.

Exercise 3. Consider the random walk $S_n = \sum_1^n X_k$, $S_0 = 0$, the X_k's being iid, $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$, $\mathcal{F}_n = \sigma(X_i, 0 \leq i \leq n)$.

Prove that $(S_n^2 - n, n \geq 0)$ is a $(\mathcal{F}_n)-$martingale. Let τ be a bounded stopping time. Prove that $\mathbb{E}(S_\tau^2) = \mathbb{E}(\tau)$.

Take now $\tau = \inf\{n \mid S_n \in \{-a, b\}\}$, where $a, b \in \mathbb{N}^*$. Prove that $\mathbb{E}(S_\tau) = 0$ and $\mathbb{E}(S_\tau^2) = \mathbb{E}(\tau)$. What is $\mathbb{P}(S_\tau = -a)$? What is $\mathbb{E}(\tau)$?

By justifying the limit $b \to \infty$, prove that the expectation of the hitting time of $-a$ is infinity.

Exercise 4. In a game between a gambler and a croupier, suppose that the total capital in play is 1. After the nth hand the proportion of the capital held by the gambler is denoted $X_n \in [0,1]$, thus that held by the croupier is $1 - X_n$. We assume $X_0 = p \in (0,1)$. The rules of the game are such that after n hands, the probability for the gambler to win the $(n+1)$th hand is X_n; if he does, he gains half of the capital the croupier held after the nth hand, while if he loses he gives half of his capital. Let $\mathcal{F}_n = \sigma(X_i, 1 \leq i \leq n)$.

a) Show that $(X_n)_{n \geq 0}$ is a $(\mathcal{F}_n)_{n \geq 0}$ martingale.

b) Show that $(X_n)_{n \geq 1}$ converges a.s. and in L^2 towards a limit Z.

c) Show that $\mathbb{E}(X_{n+1}^2) = \mathbb{E}(3X_n^2 + X_n)/4$. Deduce that $\mathbb{E}(Z^2) = \mathbb{E}(Z) = p$. What is the law of Z?

d) For any $n \geq 0$, let $Y_n = 2X_{n+1} - X_n$. Find the conditional law of X_{n+1} knowing $\mathcal{F}_n$. Prove that $\mathbb{P}(Y_n = 0 \mid \mathcal{F}_n) = 1 - X_n$, $\mathbb{P}(Y_n = 1 \mid \mathcal{F}_n) = X_n$ and express the law of Y_n.

e) Let $G_n = \{Y_n = 1\}$, $L_n = \{Y_n = 0\}$. Prove that $Y_n \to Z$ a.s. and deduce that $\mathbb{P}(\liminf_{n\to\infty} G_n) = p$, $\mathbb{P}(\liminf_{n\to\infty} L_n) = 1 - p$. Are the variables $\{Y_n, n \geq 1\}$ independent ?

f) Interpret the questions c), d), e) in terms of gain, loss, for the gambler.

Exercise 5 (bonus). Let X be a standard random walk in dimension 1, and for any positive integer a, $\tau_a = \inf\{n \geq 0 \mid X_n = a\}$. For any $\theta > 0$, calculate

$$\mathbb{E}\left((\cosh\theta)^{-\tau_a}\right).$$

Hint: look for a pertinent martingale of exponential type.