Stochastic calculus, homework 6, due November 14th.

Exercise 1. Let B be a Brownian motion and $T_a = \inf\{s \geq 0 : B_s = a\}$. Prove (with no calculation) that for any $b > a > 0$ the random variable $T_b - T_a$ is independent of T_a.

Simulate a Brownian motion (based on the method suggested by Donsker's theorem, with a thousand time steps between times 0 and 1) and the corresponding process $(T_a)_{a\geq 0}$. Give a printed copy of the code with the homework (no matter which language) and a samples of both curves.

Exercise 2. Let B be a Brownian motion starting at $x > 0$, and $T_0 = \inf\{s \geq 0 : B_s = 0\}$. That is the distribution of $\sup_{t\leq T_0} B_t$?

Hint: at some point in class we studied maxima of positive martingales converging to 0.

Exercise 3. Let $X_t = \int_0^t (\sin s)\mathrm{d}B_s$. Prove that this is a Gaussian process. What are $\mathbb{E}(X_t)$ and $\mathbb{E}(X_s X_t)$? Prove that

$$X_t = (\sin t)B_t - \int_0^t (\cos s)B_s \mathrm{d}s.$$

Exercise 4. Prove that if f is a deterministic continuous square integrable function,

$$\mathbb{E}\left(B_t \int_0^\infty f(s)\mathrm{d}B_s\right) = \int_0^t f(s)\mathrm{d}s.$$