Stochastic calculus, homework 9, due Tuesday December 12th.

Below B is a standard Brownian motion, adapted with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$.

Exercise 1. We consider the stochastic differential equation $\mathrm{d}X_t = \alpha X_t \mathrm{d}t + \beta X_t \mathrm{d}B_t$, $X_0 = 1$.

(i) Prove that $X_t = e^{(\alpha - \frac{\beta^2}{2})t + \beta B_t}$ is a solution.
(ii) Show that, for $\alpha \geq 0$, X is a submartingale with respect to $(\mathcal{F}_t)_{t\geq 0}$. For which α is it a martingale?

Exercise 2. We consider the stochastic differential equation $\mathrm{d}X_t = (b + \beta X_t)\mathrm{d}B_t$, $X_0 = x$ with $x \neq -b/\beta$.

(i) For any $y \neq -b/\beta$, we define $h(y) = \frac{1}{\beta}\log\left|\frac{b+\beta y}{b+\beta x}\right|$. What equation does $Y_t = h(X_t)$ satisfy?
(ii) What is the solution to the initial stochastic differential equation?

Exercise 3. We consider the stochastic differential equation $\mathrm{d}X_t = a(b - X_t)\mathrm{d}t + \sigma\sqrt{X_t}\mathrm{d}B_t$, $X_0 = x$ with $x > 0$. Assume there exists a solution in the strongest sense you want, and that this solution is as integrable as you want.

Calculate the expectation and variance of X_t

Exercise 4. We consider the stochastic differential equation $\mathrm{d}X_t = -\alpha^2 X_t^2(1 - X_t)\mathrm{d}t + \alpha X_t(1 - X_t)\mathrm{d}B_t$, $X_0 = x$ with $x \in (0, 1)$.

(i) Write a program to simulate a trajectory and show a sample plot.
(ii) Let $Y_t = X_t/(1 - X_t)$. What stochastic differential equation does Y satisfy?
(iii) Show that

$$X_t = \frac{x e^{\alpha B_t - \alpha^2 \frac{t}{2}}}{x e^{\alpha B_t - \alpha^2 \frac{t}{2}} + 1 - x}$$

is a solution.

Exercise 5. Consider the general equation

$$\mathrm{d}X_t = (c(t) + d(t)X_t)\mathrm{d}t + (e(t) + f(t)X_t)\mathrm{d}B_t,\ X_0 = 0.$$

where c, d, e, f are deterministic. We try to find a solution of type $X = X^{(1)}X^{(2)}$ where

$$\mathrm{d}X_t^{(1)} = d(t)X_t^{(1)}\mathrm{d}t + f(t)X_t^{(1)}\mathrm{d}B_t,\ X_0^{(1)} = 1,$$
$$\mathrm{d}X_t^{(2)} = a(t)\mathrm{d}t + b(t)\mathrm{d}B_t,\ X_0^{(2)} = X_0,$$

and a, b are stochastic processes to be chosen.

(i) Prove that $X_t^{(1)} = e^{\int_0^t f(s)\mathrm{d}B_s - \frac{1}{2}\int_0^t f(s)^2\mathrm{d}s + \int_0^t d(s)\mathrm{d}s}$ is a solution.
(ii) Identify necessary formulas for a and b.
(iii) Conclude a general formula for the solution of the initial equation.

Exercise 6. For a given Brownian motion B, let X be a solution of

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}B_t + b(X_t)\mathrm{d}t,\ X_0 = x,$$

and $X^{(n)}$ be a solution of

$$\mathrm{d}X_t = \sigma^{(n)}(X_t)\mathrm{d}B_t + b^{(n)}(X_t)\mathrm{d}t,\ X_0 = x,$$

where all functions are Lipschitz with the same absolute constant independent of n. Assume pointwise convergence of $\sigma^{(n)}$ to σ, and of $b^{(n)}$ to b. Prove that for any $t > 0$, as $n \to \infty$,

$$\mathbb{E}\left(\sup_{[0,t]} |X_s - X_s^{(n)}|^2\right) \to 0.$$

Exercise 7. Let B^1 and B^2 be independent Brownian motions, defined on the same probability space. Let

$$X_t = e^{B_t^1} \int_0^t e^{-B_s^1} \mathrm{d}B_s^2,\ Z_t = \sinh B_t^1.$$

Prove that both processes have the same distribution.