Stochastic calculus, midterm exam

Lecture notes are not be allowed. Exercises with a $(*)$ are probably more difficult than the others.

Exercise 1. Define what a submartingales in discrete time is.

Exercise 2. Let $(X_n)_{n\geq 1}$ be a martingale in discrete time and $S \leq T$ be two stopping times. Give an example in which $\mathbb{E}(X_T \mid \mathscr{F}_S) = X_S$ does not hold (no need for a proof).

Exercise 3. Let X be a nonnegative martingale in discrete time. Does it converge almost sureley? If yes, thanks to which theorem? If no, give a counterexample (no need for a proof).

Exercise 4. Given a filtration in discrete time, define a stopping time. Give an example. Give an example of a time which is not a stopping time (no need for a proof).

Exercise 5. Let X be a Gaussian random variable with mean μ and variance σ^2. For any real t, what is $\mathbb{E}(e^{tX})$?

Exercise 6. Let X be a standard Gaussian random variable, and ε an independent Bernoulli random variable ($\mathbb{P}(\varepsilon = 1) = \mathbb{P}(\varepsilon = -1) = 1/2$). Is $(X, \varepsilon X)$ a Gaussian vector? Justify your answer.

Exercise 7. Define a standard Brownian motion.

Exercise 8. Let B be a standard Brownian motion and $\lambda > 0$. Is $(\sqrt{\lambda} B_{\lambda t})_{t\geq 0}$ a standard Brownian motion? Justify your answer.

Exercise 9. Let B be a standard Brownian motion. For which $\alpha > 0$ does $\lim_{t\to 0^+} \frac{B_t}{t^\alpha}$ exist almost surely? Then, what is the limit? You don't need to prove anything, just answer.

Exercise 10. Let B be a standard Brownian motion. Are the following assertions right or wrong? If right, give the theorem it relies on. If wrong, give a counterexample (no proof needed, just a counterexample).

(i) If T is an almost surely finite stopping time, then $(B_{T+t} - B_T)_{t\geq 0}$ is a standard Brownian motion.
(ii) If T is an almost surely finite random time, then $(B_{T+t} - B_T)_{t\geq 0}$ is a standard Brownian motion.

Exercise 11. Let $(X_k)_{k\geq 0}$ be i.i.d. standard Gaussian random variables, $\mathscr{F}_n = \sigma(X_k, k \leq n)$, and $S_n^{(\mu)} = \sum_{k=0}^n X_k + \mu n$. For which values of the real parameter

μ is $(S_n^{(\mu)})_{n\geq 0}$ a $(\mathscr{F}_n)_{n\geq 0}$-supermartingale? Prove it.

Exercise 12. With the same notations as in exercise 11, find $c > 0$ such that $((S_n^{(0)})^2 - cn)_{n\geq 0}$ is a $(\mathscr{F}_n)_{n\geq 0}$-martingale. Prove it.

Exercise 13. With the same notations as in exercise 11, find $c > 0$ such that $(e^{S_n^{(0)} - cn})_{n\geq 0}$ is a $(\mathscr{F}_n)_{n\geq 0}$-martingale. Prove it. What does this martingale converge to, almost surely? Prove it.

Exercise 14 (*). State Donsker's theorem. With the same notations as in exercise 11, find $\alpha > 0$ such that $n^{-\alpha}\sum_{k\leq n} S_k^{(0)}$ converges as $n\to\infty$, to a non-trivial distribution. What is the density of this limit?

Exercise 15. Let B be a standard Brownian motion. Calculate $\mathbb{E}\left((\int_0^1 B_s \mathrm{d}s)^2\right)$.

Exercise 16 (*). Let B be a standard Brownian motion, $\mu\in\mathbb{R}$. Calculate $\mathbb{E}\left(\sup_{0\leq s\leq 1} e^{\mu B_s}\right)$.

Exercise 17. Let B be a standard Brownian motion. What is $\lim_{t\to\infty}\frac{B_t}{t}$, almost surely? Prove it.

Exercise 18. Let X be a $\mathscr{F}$ -measurable integrable random variable, and $\mathscr{F}_1\subset\mathscr{F}_2\subset\cdots\subset\mathscr{F}_n\subset\cdots\subset\mathscr{F}$ be a filtration. Prove that $(\mathbb{E}(X\mid\mathscr{F}_n))_{n\geq 1}$ is a $(\mathscr{F}_n)_{n\geq 1}$-martingale.

Exercise 19 (*). Let B be a standard Brownian motion and $T_1^* = \inf\{t\geq 0 \mid |B_t| = 1\}$. Prove that $\sup_{0\leq t\leq 1}|B_t|$ and $\frac{1}{\sqrt{T_1^*}}$ have the same distribution.

Exercise 20 (*). Let $B^{(1)}$ and $B^{(2)}$ be independent standard Brownian motions. Define the complex-valued process $B_t = B_t^{(1)} + \mathrm{i}B_t^{(2)}$. Let D be a straight line in the complex plane. What is the distribution of $T = \inf\{t\geq 0 : B_t\in D\}$?