Edge Universality of Beta Ensembles

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Abstract

We prove the edge universality of the beta ensembles for any $\beta \geqslant 1$, provided that the limiting spectrum is supported on a single interval, and the external potential is $\mathscr{C}^4$ and regular. We also prove that the edge universality holds for generalized Wigner matrices for all symmetry classes. Moreover, our results allow us to extend bulk universality for beta ensembles from analytic potentials to potentials in class $\mathscr{C}^4$.

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Contents

1 Introduction

Eigenvalues of random matrices were envisioned by Wigner as universal models for highly correlated systems. A manifestation of this general principle is the universality of random matrix statistics, i.e., that the eigenvalue distributions of large matrices are universal in the sense that they depend only on the symmetry class of the matrix ensemble, but not on the distributions of the matrix elements. These universal eigenvalue distributions are different for eigenvalues in the interior of the spectrum and for the extreme eigenvalues near the spectral edges. In this paper, we will focus on the edge universality.

Let λ_N be the largest eigenvalue of an $N \times N$ random Wigner matrix with normalization chosen such that the bulk spectrum is $[-2, 2]$. The probability distributions of λ_N for the classical Gaussian ensembles are identified by Tracy and Widom [56, 57] to be

$$\lim_{N\to\infty} \mathbb{P}(N^{2/3}(\lambda_N - 2) \leqslant s) = F_\beta(s),$$

where $F_\beta(s)$ can be computed in terms of Painlevé equations and $\beta = 1, 2, 4$ corresponds respectively to the classical orthogonal, unitary or symplectic ensemble. The edge universality means that the distributions of λ_N are given by F_β for non Gaussian ensembles as well. In fact, this holds not only for the largest eigenvalue, but the joint distributions of any finitely many "edge eigenvalues" are universal as well.

The edge universality for a large class of Wigner matrices was first proved via the moment method by Soshnikov [51] for unitary and orthogonal ensembles. This method requires that the distribution of the matrix elements be symmetric. The symmetry assumption was partially removed in [47, 54] and it was completely removed in [30]. In addition to the symmetry assumption, the moment method also requires that sufficient high moments of the matrix elements be finite. This assumption was greatly relaxed in [19] and it was finally proved by Lee and Yin [41] that essentially the finiteness of the fourth moment is the sufficient and necessary condition for the Tracy-Widom edge universality to hold (an almost optimal necessary condition was established earlier in [3]).

We now turn to the edge universality for invariant ensembles. These are matrix models with probability density on the space of $N \times N$ matrices H given by $Z^{-1} e^{-N\beta \mathrm{Tr} V(H)/2}$ where V is a real valued potential and $Z = Z_N$ is the normalization. The parameter β is determined by the symmetry class of H. The probability distribution of the ordered eigenvalues of H on the simplex determined by $\lambda_1 \leqslant \ldots \leqslant \lambda_N$ is given by

$$\mu^{(N)}(\mathrm{d}\boldsymbol{\lambda}) \sim e^{-\beta N \mathcal{H}(\boldsymbol{\lambda})} \mathrm{d}\boldsymbol{\lambda}, \quad \mathcal{H}(\boldsymbol{\lambda}) = \sum_{k=1}^{N} \frac{1}{2} V(\lambda_k) - \frac{1}{N} \sum_{1 \leqslant i < j \leqslant N} \log(\lambda_j - \lambda_i). \tag{1.1}$$

For classical invariant ensembles, i.e., $\beta = 1, 2, 4$, it is well-known that the correlation functions can be expressed in terms of orthogonal polynomials. Historically, they have been first analyzed in the bulk. The analysis at the edges is not a straightforward generalization of that in the bulk and serious technical hurdles had to be overcome. Nevertheless, the edge universality was proved by Deift-Gioev [12] for general polynomial potentials, by Pastur-Shcherbina [44] and Shcherbina [49] for real analytic, even potentials.

The measure μ (1.1) can also be considered for non classical values of β, i.e., $\beta \notin \{1, 2, 4\}$, although in this case there is no simple matrix ensemble behind the model. It is a Gibbs measure of particles in $\mathbb{R}$ with a logarithmic interaction and with an external potential V, where the parameter β is interpreted as the inverse temperature and can be an arbitrary positive number. We will often refer to the variables λ_j as particles or points and the system is often called the beta ensemble or log-gas (at inverse temperature β). We will use these two terminologies interchangeably. When $\beta \notin \{1, 2, 4\}$ and the potential V is general, no simple expression of the correlation functions in terms of orthogonal polynomials is known. For certain special potentials and even integer β, however, there are still explicit formulas for correlation functions [32]. Furthermore, for general β in the Gaussian case, i.e., when V is quadratic, Dumitriu and Edelman [18] proved that the measure (1.1) describes the eigenvalue distribution of a special tridiagonal matrix. Using this connection, Ramírez, Rider and Virág were able to characterize the edge distributions for all β [48]; a characterization of the bulk statistics of the Gaussian beta ensembles was obtained in [58]. A similar approach, independently of our current work, resulted in the proof of edge universality for convex polynomial potentials [38].

We now compare the notions of edge and bulk universality. The edge universality refers to the distributions of individual eigenvalues. However, according to Wigner's original vision, the bulk universality concerns *differences* of neighboring eigenvalues, i.e., gap distributions. The bulk universality is often formulated in terms of local correlation functions. These two notions are equivalent only after a certain averaging in the energy parameter. Strictly speaking, there are three notions of bulk universality: (i) in a weak sense which allows for energy averaging; (ii) correlation function universality at a fix energy; (iii) gap universality at a fixed label j. Clearly, universality in the sense (ii) or (iii) implies (i).

The bulk universality in the sense of (ii) for classical invariant ensembles was proved in [7, 13–16, 45, 46, 50, 59] using methods related to orthogonal polynomials. For Wigner ensembles, universality for Hermitian matrices in the sense (ii) was proved in [20, 21, 26, 55] and for all symmetry classes in the sense (i) in [23, 24, 30]. The gap universality, i.e., (iii), is in fact much harder to obtain; it was proved only recently in [25] both for invariant and Wigner ensembles using new ideas from parabolic regularity theory (the special case of hermitian matrices with the first four moments of the matrix elements matching those of GUE was proved earlier in [53]). The bulk universality for log-gases for general β was proved in the sense (i) and (iii) in [8, 9, 25]. The bulk universality in the sense (ii) for Wigner ensembles with $\beta \neq 2$ and for log-gases with $\beta \notin \{1, 2, 4\}$ remains open problems.

Returning to the edge universality, we will establish the following two results in this paper: (1) edge universality for $\mathscr{C}^4$ potentials and for all $\beta \geqslant 1$; (2) edge universality for generalized Wigner matrices (these are matrices with independent but not necessarily identically distributed entries, see Definition 2.6). An important ingredient of the proof will be an optimal location estimate for the particles up to the edge, for external potentials of class $\mathscr{C}^4$. This rigidity will also allow us to remove the analyticity assumption from previous results about bulk universality [8, 9, 25].

We now outline the technique used in this paper. For the edge universality of invariant ensembles, the basic idea is to consider a local version of the log-gas (1.1). This is the measure on K consecutive particles that is obtained by fixing all other particles which act as boundary conditions. Following the standard language in statistical physics, we will refer to these local measures as local log-gases. Our core result is the "uniqueness" of this local measure in the limit $K \to \infty$ assuming that the boundary conditions are "good". By uniqueness, we mean that the distributions of the particles far away from the boundaries are independent of choice of the "good" boundary conditions. This idea first appeared in [8] for proving the bulk universality of log-gases. However, the uniqueness of the local Gibbs state in the bulk was defined slightly differently in [8]; only the gap distributions were required to be independent of the boundary conditions.

It is well-known that the uniqueness of local Gibbs measures in the thermodynamical limit is closely related to the decay of correlation functions. The work of Gustavsson [34] for the special $\beta = 2$ and Gaussian case (i.e., the GUE case) indicates that in general the point-point correlation function decays only logarithmically in the bulk, i.e., $\langle \lambda_i; \lambda_j \rangle_{\mu_\beta} \sim \log |i-j|$. Gibbs measures with such a slow decay are typically not unique in the usual sense. The key reason why we were able to prove the uniqueness of the gap distributions of local log-gases in the bulk [25] is the observation that the point-gap correlation, $\langle \lambda_i; \lambda_j - \lambda_{j+1} \rangle_{\mu_\beta} \sim \partial_j \langle \lambda_i; \lambda_j \rangle_{\mu_\beta}$ is expected to decay much faster due to the simple reason that $\partial_j \log |i-j| \sim 1/|j|$. In real statistical physics system, however, it is very difficult to compute derivatives of correlation functions unless they are expressed almost explicitly by some expansion method. The *Dirichlet form inequality* [23], a main tool in [8], allows us to take advantage of the fact that the observables are functions of the gaps.

In the subsequent work [25], the correlation functions were expressed in terms of off-diagonal matrix elements of heat kernels describing random walks in random environments. This representation in a lattice setting was given in [17, 33]. In a slightly different formulation it already appeared in the earlier paper of Naddaf and Spencer [42], which was a probabilistic formulation of the idea of Helffer and Sjöstrand [35]. Using this representation, the decay of the point-gap correlation amounts to the Hölder continuity of the heat kernel for the random walk dynamics [25]. We note that the jump rates in this random walk dynamics are long ranged and contain short distance singularities depending on the stochastically driven environment. The proof of the Hölder continuity in [25] requires extending the De Giorgi-Nash-Moser type method of Caffarelli, Chan and Vasseur [11] to the singular coefficient case and providing a priori estimates such as rigidity and level repulsion.

It should be stressed that, despite these efforts, only gap distributions but not those of individual eigenvalues were identified in [25]. Edge universality, however, is exactly about individual eigenvalues and not about gaps. The surprising fact is that correlation functions of log-gases decay as a power law near the edges! Thus we do not need the Hölder regularity argument from [25] to analyze the edges. Instead, in this paper we rely on the energy method from parabolic PDE's and on certain new Sobolev type inequalities for nonlocal operators to prove the decay of off-diagonal elements of the heat kernel. For this purpose, we will need rigidity and level repulsion estimates near the edges. We will extend the *multi-scale analysis of the loop equation*, first appeared in [8], in two directions. First, this analysis will be performed along the whole spectrum, including the edge, where the change of scaling poses a major difficulty; second, analyticity of the external potential is not required, thanks to a new analysis of the loop equation.

For the edge universality of Wigner ensembles, we will use the idea of *the local relaxation flow* initiated in [23, 24] and the Green function comparison theorem from [29]. This theorem can be used for both the bulk or the edge universality. In particular, the eigenvalue distributions of two Wigner ensembles near the edges are the same provided that the variances of the matrix elements of the two ensembles are identical. This implied the edge universality for Wigner matrices [30].

On the other hand, if the variances of the matrix elements are allowed to vary, then the matrix cannot be matched to a Gaussian Wigner matrix. Thus the edge universality for generalized Wigner matrices cannot be proved directly with the Green function comparison theorem. Using the uniqueness of *local* log-gases, we can identify the distributions of the edge particles in the Dyson Brownian Motion (DBM). This implies the edge universality for general classes of Gaussian divisible ensembles with varying variance. Finally, we will use the Green function comparison theorem to bridge the gap between generalized Wigner matrices and their Gaussian divisible counterparts.

We emphasize that the uniqueness of local log-gases plays a *central* role both in the edge universality of log-gases and in our analysis of edge points in DBM. For log-gases, it is natural to localize the problem so that the external potential can be replaced by its first order approximation and thus it becomes universal after scaling. However, localization of the measure in general introduces very large errors in strongly correlated systems. The key observation is that there are strong cancellations in the effective potential for "good" boundary conditions. The significance of the local log-gases in the proof of proof of universality for Wigner matrices is subtler, and will be explained in details in Section 5.

Convention. We use the letters C, c to denote positive constants, independent of N, whose values may change from line to line. We will often estimate the probability of rare events $\Omega = \Omega_N$ that are typically either subexponentially small, $\mathbb{P}(\Omega) \leqslant \exp(-N^c)$ or small by an N-power; $\mathbb{P}(\Omega) \leqslant N^{-c}$. In both cases it is understood that the statements hold for any sufficiently large $N \geqslant N_0$. We will not follow the precise values of the exponents c or the thresholds N_0.

2 Main results

We will have two related results, one concerns the generalized Wigner ensembles, the other one the general beta ensembles.

2.1 Edge universality of the beta ensembles

We first define the beta ensembles. Let $\Xi^{(N)} \subset \mathbb{R}^N$ denote the set

$$\Xi^{(N)} := \{\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_N) \ : \ \lambda_1 \leqslant \lambda_2 \leqslant \dots \leqslant \lambda_N\}. \tag{2.1}$$

Consider the probability distribution on Ξ_N given by

$$\mu_{\beta,V}^{(N)} = \mu^{(N)}(\mathrm{d}\boldsymbol{\lambda}) = \frac{1}{Z_{\beta,V}^{(N)}} e^{-\beta N \mathcal{H}(\boldsymbol{\lambda})} \mathrm{d}\boldsymbol{\lambda}, \qquad \mathcal{H}(\boldsymbol{\lambda}) = \sum_{k=1}^N \frac{1}{2} V(\lambda_k) - \frac{1}{N} \sum_{1 \leqslant i < j \leqslant N} \log(\lambda_j - \lambda_i), \tag{2.2}$$

where $Z_{\beta,V}^{(N)}$ is the normalization. In the following, we often omit the parameters N, β and V in the notation and we will write μ for $\mu^{(N)}$. Sometimes emphasize the dependence in the external potential by writing $\mu = \mu_V$. We will use $\mathbb{P}^\mu$ and $\mathbb{E}^\mu$ to denote the probability and the expectation with respect to μ.

We will view μ as a Gibbs measure of N particles in $\mathbb{R}$ with a logarithmic interaction, where the parameter $\beta > 0$ is interpreted as the inverse temperature. We will refer to the variables λ_j as particles or points and the system is called *log-gas* or *general beta ensemble*. We will assume that the potential V is a $\mathscr{C}^4$ real function in $\mathbb{R}$ such that its second derivative is bounded below, i.e., we have

$$\inf_{x \in \mathbb{R}} V''(x) \geqslant -2W \tag{2.3}$$

for some constant $W \geqslant 0$, and

$$V(x) > (2 + \alpha) \ln(1 + |x|), \tag{2.4}$$

for some $\alpha > 0$, if $|x|$ is large enough. It is known [10] that under these (in fact, even weaker) conditions the measure is normalizable, $Z^{(N)} < \infty$. Moreover, the averaged density of the empirical spectral measure, defined as

$$\varrho_1^{(N)}(\lambda) = \varrho_1^{(N,\beta,V)}(\lambda) := \mathbb{E}^\mu \frac{1}{N} \sum_{j=1}^N \delta(\lambda - \lambda_j), \qquad \lambda \in \mathbb{R},$$

converges weakly to a continuous function $\varrho = \varrho_V$, the equilibrium density, with compact support. We assume that ϱ is supported on a single interval $[A, B]$, and that V is *regular* in the sense of [39]. We recall that V is regular if its equilibrium density ϱ is positive on (A,B) and vanishes like a square root at each of the endpoints of $[A,B]$, that is

$$\begin{aligned}\varrho(t) &= s_A\sqrt{t-A}\,(1+\mathrm{O}\,(t-A))\,,\ t\to A^+,\\ \varrho(t) &= s_B\sqrt{B-t}\,(1+\mathrm{O}\,(B-t))\,,\ t\to B^-,\end{aligned} \tag{2.5}$$

for some constants $s_A,\ s_B>0$. We remark that this regularity assumption is not a strong constraint; regular potentials V form a dense and open subset in the space of the potentials with a natural topology [39].

Let the *limiting classical location* of the k-th particle, $\gamma_k=\gamma_k(N)$, be defined by

$$\int_{-\infty}^{\gamma_k}\varrho(s)\mathrm{d}s=\frac{k}{N}. \tag{2.6}$$

Finally, we introduce the notation $[\![p,q]\!]=[p,q]\cap\mathbb{Z}$ for any real numbers $p<q$.

We will be interested in the usual n-point correlation functions, generalizing $\varrho_1^{(N)}$, and defined by

$$\varrho_n^{(N)}(\lambda_1,\dots,\lambda_n)=\int_{\mathbb{R}^{N-n}}\mu^{\#}(\boldsymbol{\lambda})\mathrm{d}\lambda_{n+1}\dots\mathrm{d}\lambda_N, \tag{2.7}$$

where $\mu^{\#}$ is the symmetrized version of μ given in (2.2) but defined on $\mathbb{R}^N$ instead of the simplex $\Xi^{(N)}$:

$$\mu^{\#(N)}(\mathrm{d}\boldsymbol{\lambda})=\frac{1}{N!}\mu(\mathrm{d}\boldsymbol{\lambda}^{(\sigma)}),$$

where $\boldsymbol{\lambda}^{(\sigma)}=(\lambda_{\sigma(1)},\dots,\lambda_{\sigma(N)})$, with $\lambda_{\sigma(1)}<\dots<\lambda_{\sigma(N)}$. Our main result is the following.

In the following theorem, we consider two regular potentials, V and $\widetilde{V}$, such that their equilibrium densities ϱ_V and $\varrho_{\widetilde{V}}$ are supported on a single interval. Without loss of generality (by applying a simple scaling and shift), we may also assume that the singularities at the left edge match and both occur at $A=0$, with the same constant $s_A=1$:

$$\varrho_V(t)=\sqrt{t}\,(1+\mathrm{O}\,(t))\,,\ \varrho_{\widetilde{V}}(t)=\sqrt{t}\,(1+\mathrm{O}\,(t))\,,\ t\to 0^+. \tag{2.8}$$

Theorem 2.1 (Edge universality for beta ensembles)**.** *Let $\beta\geqslant 1$ and V, $\widetilde{V}$ be $\mathscr{C}^4$, regular and satisfy (2.3), (2.4). Assume that the equilibrium density ϱ_V and $\varrho_{\widetilde{V}}$ are supported on a single interval and satisfy (2.8).*

For any constant $\kappa<2/5$ there exists $\chi>0$ such that the following holds. Take any fixed $m\geqslant 1$ and a continuously differentiable compactly supported function $O:\mathbb{R}^m\to\mathbb{R}$. There exists a constant $C>0$ such that for any N and $\Lambda\subset[\![1,N^\kappa]\!]$ with $|\Lambda|=m$, we have

$$\left|(\mathbb{E}^{\mu_V}-\mathbb{E}^{\mu_{\tilde{V}}})O\left(\left(N^{2/3}j^{1/3}(\lambda_j-\gamma_j)\right)_{j\in\Lambda}\right)\right|\leqslant CN^{-\chi}. \tag{2.9}$$

Remark. Note that one may define γ_j in (2.6) with respect to the measure ϱ_V or $\varrho_{\widetilde{V}}$, it does not make any difference in the above theorem when $\kappa<2/5$: from (2.8) one obtains $\gamma_j-\widetilde{\gamma}_j=\mathrm{O}\left((j/N)^{4/3}\right)$, which is of smaller order than the scale $N^{-2/3}j^{-1/3}$ detected in (2.9). We also remark that Theorem 2.1 is formulated for points near the lower spectral edge A, but a similar statement holds near the upper spectral edge B.

The first results on edge universality for invariant ensembles concerned the classical values of $\beta=1,2,4$. The case $\beta=2$ and real analytic V was solved in [12,15]. The $\beta=1,4$ cases are considerably harder than

$\beta = 2$. For $\beta = 1, 4$ universality was first solved for polynomial potentials in [12], then the real analytic case for $\beta = 1$ in [44, 49], which also give an alternative proof for $\beta = 2$. Finally, independently of our work with a completely different method, edge universality for any $\beta > 0$ and convex polynomial V was recently proved in [38].

Choosing $S = [\![1, m]\!]$ and $\widetilde{V}(x) = x^2$ in the previous theorem allows us to identify the universal distribution from Theorem 2.1 with the Tracy-Widom distribution with parameter $\beta > 0$. This distribution can be represented via the *stochastic Airy operator*. We refer to [48] for its proper definition, the Hilbert space it acts on, and the proof that its smallest eigenvalues describe the asymptotic edge fluctuations of the Gaussian beta ensembles.

Corollary 2.2 (Identification of the edge distribution)**.** *Let $\beta \geqslant 1$ and $m \in \mathbb{N}$ be fixed, and $\Lambda_1 < \cdots < \Lambda_m$ the m smallest eigenvalues of the stochastic Airy operator $-\partial_{xx} + x + \frac{2}{\sqrt{\beta}} b'_x$ on $\mathbb{R}_+$, where b'_x is a white noise. Let V be $\mathscr{C}^4$, regular with equilibrium density supported on a single interval, and satisfy (2.3), (2.4), (2.8). Then the following convergence in distribution holds:*

$$(N/2)^{2/3}(\lambda_1 - A, \dots, \lambda_m - A) \to (\Lambda_1, \dots, \Lambda_m).$$

Theorem 2.1 can be used to show Gaussian fluctuations for the points in an intermediate distance from the edge. Indeed, such fluctuations were proved by Gustavsson in [34] in the $\beta = 2$ Gaussian case (GUE) for all eigenvalues, and this was extended $\beta = 1$ and 4 in [43]. Combining these results with Theorem 2.1 immediately gives the following statement (here $k \sim N^{\vartheta}$ means $\log k / \log N \to \vartheta$).

Corollary 2.3 (Gaussian fluctuations)**.** *Let $\beta = 1, 2$ or 4 and the potential V be $\mathscr{C}^4$, regular such that the equilibrium density ϱ_V is supported on a single interval and satisfies (2.8). Consider the measure $\mu^{(N)}_{\beta,V}$. We define*

$$X_i = c\ \frac{\lambda_i - \gamma_i}{(\log i)^{1/2} N^{-2/3} i^{-1/3}},$$

where $c = (3/2)^{1/3} \pi \beta^{1/2}$. Fix $\kappa < 2/5$. Then for any sequence $i = i_N \to \infty$, with $i \leqslant N^{\kappa}$, we have $X_i \to \mathcal{N}(0, 1)$ in distribution.

Moreover, for some fixed $m > 0$ and $\delta \in (0, 2/5)$, let $k_1 < \cdots < k_m$ satisfy $k_1 \sim N^{\delta}$, and $k_{i+1} - k_i \sim N^{\vartheta_i}$, $0 < \vartheta_i < \delta$. Then $(X_{k_1}, \dots, X_{k_m})$ converges to a Gaussian vector with covariance matrix $\Lambda_{ij} = 1 - \delta^{-1} \max\{\vartheta_k, i \leqslant k < j\}$ if $i < j$, $\Lambda_{ii} = 1$.

We note that if Gustavsson's result on Gaussian fluctuations were known for the general Gaussian beta ensembles, then this corollary would prove a central limit theorem for general beta ensembles near the edge.

An important element in the proof of Theorem 2.1 consists in proving the following rigidity estimate asserting that any particle λ_k is very close to its limiting classical location. For any $k \in [\![1, N]\!]$ we define

$$\widehat{k} := \min\{k, N + 1 - k\}.$$

For orientation, we note that

$$\gamma_k \sim \left(\frac{\widehat{k}}{N}\right)^{2/3}, \qquad \gamma_{k+1} - \gamma_k \sim N^{-2/3} (\widehat{k})^{-1/3},$$

where $A \sim B$ means $c \leqslant A/B \leqslant C$. More precisely, by the square-root singularity of ϱ near the left edge,

$$\gamma_k \sim \left(\frac{k}{N}\right)^{2/3} \Big[1 + O\big((k/N)^{2/3}\big)\Big], \qquad \gamma_{k+1} - \gamma_k \sim N^{-2/3} k^{-1/3} \Big[1 + O\big((k/N)^{2/3}\big)\Big], \tag{2.10}$$

and similar asymptotics hold near the right edge. The following theorem states that all particles will be close to their classical locations on this scale, up to a factor N^{ξ} with an arbitrary small exponent $\xi > 0$. Following [30], we will call such a precise bound on the locations of particles a *rigidity estimate*. The rigidity estimate in some weaker forms has already been used as a fundamental input [24] to prove the universality for Wigner matrices. It also played a key role in the proof of the bulk universality for the log-gases in [8]. The following result extends the rigidity estimate from the bulk to the edges, and removes the analyticity assumption.

Theorem 2.4 (Rigidity estimate for global measures)**.** *Let $\beta > 0$, V be $\mathscr{C}^4$, regular with equilibrium density supported on a single interval $[A, B]$, and satisfy (2.3), (2.4). For any $\xi > 0$, there are constants $c > 0$ and N_0 such that for any $N \geqslant N_0$ and $k \in [\![1, N]\!]$ we have*

$$\mathbb{P}^{\mu}\left(|\lambda_k - \gamma_k| > N^{-\frac{2}{3}+\xi}(\hat{k})^{-\frac{1}{3}}\right) \leqslant e^{-N^c}. \tag{2.11}$$

Related bounds on the concentration of the empirical density on a scale far from the optimal one (2.11) were established previously [6, 36, 50], see also references in [2].

Thanks to Theorem 2.4, bulk universality holds for beta ensembles, as stated in [8, 9, 25], without the analyticity assumption.

Theorem 2.5 (Bulk universality)**.** *Let V be $\mathscr{C}^4$, regular with equilibrium density supported on a single interval $[A, B]$, and satisfy (2.3), (2.4). Then the following two results hold.*

(i) **Correlation functions.** *For any fixed $\beta > 0$, $E \in (A, B)$, $|E'| < 2$, $n \in \mathbb{N}$ and $0 < k \leqslant \frac{1}{2}$ there exists a $\chi > 0$ such that for any continuously differentiable O with compact support we have (setting $s := N^{-1+k}$)*

$$\Bigg|\int \mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_n\, O(\alpha_1, \dots, \alpha_n)\Bigg[\int_{E-s}^{E+s} \frac{\mathrm{d}x}{2s} \frac{1}{\varrho(E)^n}\, \varrho_n^{(N)}\Big(x + \frac{\alpha_1}{N\varrho(E)}, \dots, x + \frac{\alpha_n}{N\varrho(E)}\Big)$$
$$- \int_{E'-s}^{E'+s} \frac{\mathrm{d}x}{2s} \frac{1}{\varrho_{sc}(E')^n}\, \varrho_{\mathrm{Gauss},n}^{(N)}\Big(x + \frac{\alpha_1}{N\varrho_{sc}(E')}, \dots, x + \frac{\alpha_n}{N\varrho_{sc}(E')}\Big)\Bigg]\Bigg| \leqslant CN^{-\chi}.$$

Here $\varrho_{sc}(E) = \frac{1}{2\pi}\sqrt{4 - E^2}$ is the Wigner semicircle law and $\varrho_{\mathrm{Gauss},n}^{(N)}$ are the correlation functions of the Gaussian β-ensemble, i.e. with $V(x) = x^2$.

(ii) **Gaps.** *For any fixed $\beta \geqslant 1$ and $\alpha > 0$, there is some $\varepsilon > 0$ such that for any $n \in \mathbb{N}$ and any differentiable O with compact support, and any $k, m \in [\![\alpha N, (1-\alpha)N]\!]$, we have*

$$\Big|\mathbb{E}^{\mu} O\left(Nc_k^{\mu}(\lambda_k - \lambda_{k+1}), \dots, Nc_k^{\mu}(\lambda_k - \lambda_{k+n})\right)$$
$$- \mathbb{E}^{\mathrm{Gauss}} O\left(Nc_m^{\mathrm{Gauss}}(\lambda_m - \lambda_{m+1}), \dots, Nc_m^{\mathrm{Gauss}}(\lambda_m - \lambda_{m+n})\right)\Big| \leqslant CN^{-\varepsilon}$$

where $c_k^{\mu} = \varrho^{(\mu)}(\gamma_k)$ and $c_m^{\mathrm{Gauss}} = \varrho_{sc}(\gamma_m^{\mathrm{Gauss}})$ with $\gamma_m^{\mathrm{Gauss}}$ being the m-th quantile of the semicircle law defined by $\int_{-2}^{\gamma_m^{\mathrm{Gauss}}} \varrho_{sc}(x)\mathrm{d}x = m/N$.

Proof. Part (i) was proved in [9] under the assumption that V is analytic, a hypothesis that was only required for proving rigidity in the bulk of the spectrum. Theorem 2.4 proves that V of class $\mathscr{C}^4$ is sufficient for rigidity, and the proof of the uniqueness of the Gibbs measure is identical to [8, 9]. The result in these papers were stated in a limiting form, as $N \to \infty$, and for smooth observables O, but the proofs hold for any continuously differentiable O and with an effective error bound of order $N^{-\chi}$ with some $\chi > 0$ as well. The statement (ii) holds for the same reason, being previously proved for analytic V in [25]. □

We finally remark that while the rigidity estimate (2.11) holds for any $\beta > 0$, the edge universality in Theorem 2.1 was stated only for $\beta \geqslant 1$. This restriction is mainly due to that the DBM dynamics (8.1) is known to be well-posed only for $\beta \geqslant 1$. We believe that this restriction can be removed, but we will not pursue this issue in this paper.

2.2 Edge universality of the generalized Wigner matrices

We now define the generalized Wigner ensembles. Let $H = (h_{ij})_{i,j=1}^N$ be an $N \times N$ complex Hermitian or real symmetric matrix where the matrix elements $h_{ij} = \bar{h}_{ji}$, $i \leqslant j$, are independent random variables given by a probability measure ν_{ij} with mean zero and variance $\sigma_{ij}^2 \geqslant 0$;

$$\mathbb{E}\, h_{ij} = 0, \qquad \sigma_{ij}^2 := \mathbb{E}|h_{ij}|^2. \tag{2.12}$$

The distribution ν_{ij} and its variance σ_{ij}^2 may depend on N, but we omit this fact in the notation. We also assume that the normalized matrix elements satisfy a uniform subexponential decay,

$$\mathbb{P}(|h_{ij}| > x\sigma_{ij}) \leqslant \vartheta^{-1} \exp\left(-x^{\vartheta}\right), \qquad x > 0, \tag{2.13}$$

with some fixed constant ϑ, uniformly in N, i, j.

Definition 2.6. *[29] The matrix ensemble H defined above is called generalized Wigner matrix if the following assumptions hold on the variances of the matrix elements* (2.12)

(A) *For any j fixed*

$$\sum_{i=1}^N \sigma_{ij}^2 = 1.$$

(B) *There exist two positive constants, C_1 and C_2, independent of N such that*

$$\frac{C_1}{N} \leqslant \sigma_{ij}^2 \leqslant \frac{C_2}{N}.$$

For Hermitian ensembles, we additionally assume that for each i, j the 2×2 covariance matrix

$$\Sigma_{ij} \ = \ \begin{pmatrix} \mathbb{E}(\operatorname{Re} h_{ij})^2 & \mathbb{E}(\operatorname{Re} h_{ij})(\operatorname{Im} h_{ij}) \\ \mathbb{E}(\operatorname{Re} h_{ij})(\operatorname{Im} h_{ij}) & \mathbb{E}(\operatorname{Im} h_{ij})^2 \end{pmatrix}$$

satisfies

$$\Sigma_{ij} \ \geqslant \ \frac{C_1}{N}$$

in matrix sense.

Let $\mathbb{P}^H$ and $\mathbb{E}^H$ denote the probability and the expectation with respect to this ensemble. Our result asserts that the local statistics on the edge of the spectrum are universal for any general Wigner matrix, in particular they coincide with those of the corresponding standard Gaussian ensemble.

Theorem 2.7 (Edge universality of generalized Wigner matrices)**.** *Let H be a generalized Wigner ensemble with subexponentially decaying matrix elements,* (2.13)*. For any $\kappa < 1/4$, there exists $\chi > 0$ such that the following result holds. Take any fixed $m \geqslant 1$ and a smooth compactly supported function $O : \mathbb{R}^m \to \mathbb{R}$. Then there is a constant $C > 0$ such that for any N and $\Lambda \subset [\![1, N^\kappa]\!]$ with $|\Lambda| = m$, we have*

$$\left|(\mathbb{E}^H - \mathbb{E}^{\mu_G})O\left(\left(N^{2/3}j^{1/3}(\lambda_j - \gamma_j)\right)_{j\in\Lambda}\right)\right| \leqslant C\ N^{-\chi},$$

where μ_G is the standard Gaussian GOE or GUE ensemble, depending on the symmetry class of H (It is well-known that μ_G is also given by (2.2) *with potential $V(x) = \frac{1}{2}x^2$ and with the choice $\beta = 1, 2$, respectively).*

This theorem immediately implies analogues of Corollaries 2.2 and 2.3 in the case of symmetric or Hermitian generalized Wigner ensembles.

Edge universality for Wigner matrices was first proved in [51] assuming symmetry of the distribution of the matrix elements and finiteness of all their moments. In the consequent works, after partial results in [47, 54], the symmetry condition was completely eliminated [30]. The moment condition was improved in [3, 19] and the optimal result was obtained in [41]. All these works heavily rely on the fact that the variances of the matrix elements are identical. The main point of Theorem 2.7 is to consider generalized Wigner matrices, i.e., matrices with non-constant variances. In fact, it was shown in [30] that the edge statistics for any generalized Wigner matrix are universal in the sense that they coincide with those of a generalized Gaussian Wigner matrix with the same variances, but it was not shown that the statistics are independent of the variances themselves. Theorem 2.7 provides this missing step and thus it proves the edge universality in the broadest sense.

3 Local equilibrium measures

Recall that the support of the equilibrium density ϱ was denoted by $[A, B]$. Without loss of generality, by a shift we set $A = 0$ and we will study the particles near the lower edge of the support. Fix a small exponent δ and a parameter $K = K_N$ satisfying

$$N^\delta \leqslant K \leqslant N^{1-\delta}. \tag{3.1}$$

Denote by $I = [\![1, K]\!]$ the set of the first K indices. We will distinguish the first K particles from the rest by renaming them as

$$(\lambda_1, \lambda_2, \ldots, \lambda_N) = (x_1, \ldots, x_K, y_{K+1}, \ldots y_N) \in \Xi^{(N)}.$$

Note that the particles keep their original indices. We recall the notation $\Xi^{(N)}$ for the simplex (2.1). In short we will write

$$\mathbf{x} = (x_1, \ldots, x_K) \in \Sigma^{(K)}, \qquad \text{and} \qquad \mathbf{y} = (y_{K+1}, \ldots, y_N) \in \Sigma^{(N-K)}.$$

These points are always listed in increasing order and we will refer to the y's as the *external* points and to the x's as *internal* points. We will fix the external points (often called boundary conditions) and study the conditional measures on the internal points. Note that for any fixed $\mathbf{y} \in \Xi^{(N-K)}$, all x_j's lie in the *open configuration interval*, denoted by

$$J = J_{\mathbf{y}} = (-\infty, y_{K+1}) =: (-\infty, y_+].$$

Define the *local equilibrium measure* (or *local measure* in short) on J^K with boundary condition $\mathbf{y}$ by

$$\mu_{\mathbf{y}}(\mathrm{d}\mathbf{x}) = \frac{1}{Z_{\mathbf{y}}} e^{-\beta N \mathcal{H}_{\mathbf{y}}(\mathbf{x})} \mathrm{d}\mathbf{x}, \qquad \mathbf{x} \in J^K,$$

where we introduced the Hamiltonian

$$\begin{aligned} \mathcal{H}_{\mathbf{y}}(\mathbf{x}) :=& \frac{1}{2} \sum_{i \in I} V_{\mathbf{y}}(x_i) - \frac{1}{N} \sum_{\substack{i,j \in I \\ i<j}} \log |x_j - x_i|, \\ V_{\mathbf{y}}(x) :=& V(x) - \frac{2}{N} \sum_{j \notin I} \log |x - y_j|. \end{aligned}$$

Here $V_{\mathbf{y}}(x)$ can be viewed as the external potential of a log-gas of the points $\{x_i : i \in I\}$. Although this is the natural local measure, it does not have good uniform convexity in the regime $x_1 \ll 0$. It is more convenient to consider the following modified measure σ and its local version $\sigma_{\mathbf{y}}$. For the proof of the universality of the original measure μ it will actually be sufficient to consider only the local measure $\sigma_{\mathbf{y}}$.

We will fix a small parameter $\xi > 0$ whose actual value is immaterial; it will be used to provide an multiplicative error bar of size $N^{C\xi}$ in various estimates on the location of the particles. We will not carry ξ in the notation and at the end of the proof it can be chosen sufficiently small, depending on all other exponents along the argument.

We introduce a confined measure by adding an extra quadratic potential Θ to prevent the x_i's from deviating far in the left direction:

$$\begin{aligned} \sigma(\mathrm{d}\mathbf{x}) &:= \frac{Z}{Z^\sigma} e^{-2\beta \sum_{i=1}^N \Theta\left(N^{\frac{2}{3}-\xi} x_i\right)} \mu(\mathrm{d}\mathbf{x}) = \frac{1}{Z^\sigma} e^{-\beta N \mathcal{H}^\sigma(\mathbf{x})} \mathrm{d}\mathbf{x}, \\ \mathcal{H}^\sigma(\mathbf{x}) &:= \mathcal{H}(\mathbf{x}) + \frac{2}{N} \sum_{i=1}^N \Theta\left(N^{\frac{2}{3}-\xi} x_i\right), \quad \Theta(u) = (u+1)^2 \mathbb{1}\{u < -1\}. \end{aligned} \tag{3.2}$$

The local version of the measure σ is defined in the obvious way,

$$\sigma_{\mathbf{y}}(\mathrm{d}\mathbf{x}) := \frac{1}{Z_{\mathbf{y}}^\sigma} e^{-\beta N \mathcal{H}_{\mathbf{y}}^\sigma(\mathbf{x})} \mathrm{d}\mathbf{x}, \quad \mathcal{H}_{\mathbf{y}}^\sigma(\mathbf{x}) := \mathcal{H}_{\mathbf{y}}(\mathbf{x}) + \frac{2}{N} \sum_{i \in I} \Theta\left(N^{\frac{2}{3}-\xi} x_i\right). \tag{3.3}$$

For technical reasons we will also need the following variants of σ and $\sigma_{\mathbf{y}}$ where we added slightly less convexity through Θ:

$$\begin{aligned} \widehat{\sigma}(\mathrm{d}\mathbf{x}) &:= \frac{Z}{\widehat{Z}^c} e^{-\beta \sum_{i \in I} \Theta\left(N^{\frac{2}{3}-\xi} x_i\right)} \mu(\mathrm{d}\mathbf{x}), \\ \widehat{\sigma}_{\mathbf{y}}(\mathrm{d}\mathbf{x}) &:= \frac{1}{\widehat{Z}_{\mathbf{y}}^c} e^{-\beta N \widehat{\mathcal{H}}_{\mathbf{y}}^\sigma(\mathbf{x})} \mathrm{d}\mathbf{x}, \qquad \widehat{\mathcal{H}}_{\mathbf{y}}^\sigma(\mathbf{x}) := \mathcal{H}_{\mathbf{y}}^\sigma + \frac{1}{N} \sum_{i \in I} \Theta\left(N^{\frac{2}{3}-\xi} x_i\right). \end{aligned}$$

The measures $\sigma, \widehat{\sigma}$ and their local versions depend on the parameters V, β, K and ξ but we do not carry this dependence in the notation.

Rigidity estimates proved for the global measure μ (Theorem 2.4) also hold for the local measures $\sigma_{\mathbf{y}}$ provided $\mathbf{y}$ lies in the set of *"good" boundary conditions* that is defined as follows:

$$\mathcal{R} = \mathcal{R}_K = \mathcal{R}_{K,V,\beta}(\xi) := \{\mathbf{y} \ : \ |y_k - \gamma_k| \leqslant N^{-2/3+\xi} \hat{k}^{-1/3}, \ k \notin I\}. \tag{3.4}$$

The rigidity exponent ξ will always be chosen much smaller than the exponent δ in (3.1). This guarantees that the typical length of the configuration interval, $|J| \sim \gamma_K - \gamma_1 \geqslant c(K/N)^{2/3}$, be bigger than the largest rigidity precision, $N^{-\frac{2}{3}+\xi}$.

We will need the following two modifications of $\mathcal{R}$. The first one requires that x_k be good in an expectation sense w.r.t. $\sigma_{\mathbf{y}}$, and that x_1 is not too negative. Thus we define the set

$$\mathcal{R}^* = \mathcal{R}^*_{K,V,\beta}(\xi) := \{\mathbf{y} \in \mathcal{R}_K(\xi)\ : \forall k \in I,\ |\mathbb{E}^{\sigma_{\mathbf{y}}} x_k - \gamma_k| \leqslant N^{-\frac{2}{3}+\xi} k^{-\frac{1}{3}},\ \mathbb{P}^{\widehat{\sigma}_{\mathbf{y}}}(x_1 \geqslant \gamma_1 - N^{-\frac{2}{3}+\xi}) \geqslant 1/2\}. \tag{3.5}$$

Notice that for technical reasons to be clear later on the constraint on x_1 is w.r.t. the measure $\widehat{\sigma}_{\mathbf{y}}$. This condition will be important in Sect. 6.5.

Another modification adds the condition of a level repulsion near the boundary, i.e., we define

$$\mathcal{R}^\# = \mathcal{R}^\#_{K,V,\beta}(\xi) := \{\mathbf{y} \in \mathcal{R}_{K,V,\beta}(\xi/3)\ :\ |y_{K+1} - y_{K+2}| \geqslant N^{-2/3-\xi} K^{-1/3}\}. \tag{3.6}$$

In the following theorems we establish rigidity and level repulsion estimates for the local log-gas $\sigma_{\mathbf{y}}$ with good boundary conditions $\mathbf{y}$ up to the spectral edges. These theorems extend similar estimates for the local measure $\mu_{\mathbf{y}}$ in the bulk of the spectrum established in [25] to the edges for the measure $\sigma_{\mathbf{y}}$.

Theorem 3.1 (Rigidity estimate for local measures)**.** *Fix $\beta, \xi > 0$ and, using the above notations, assume that $\mathbf{y} \in \mathcal{R}^*_K(\xi)$. Then there exists constants $C, c > 0$ (independent of $\mathbf{y}, K$) such that for large enough N we have, for any $k \in I$, and $u > 0$,*

$$\mathbb{P}^{\sigma_{\mathbf{y}}}\left(|x_k - \gamma_k| > C N^{-\frac{2}{3}+\xi} k^{-\frac{1}{3}} u\right) \leqslant e^{-cu^2}. \tag{3.7}$$

As a side comment we remark that the Gaussian decay in (3.7) is an artifact of the additional confinement in the local measure $\sigma_{\mathbf{y}}$. For the measures μ or $\mu_{\mathbf{y}}$, the tail probability of x_1 has a slower decay $\exp[-C(\gamma_1 - x_1)^{3/2}]$ in the regime $x_1 \ll \gamma_1$ in accordance with the tail behaviour of the Tracy-Widom law (for the Gaussian beta ensemble, see [40] for a detailed analysis of the edge tail behavior). However, Theorem 3.3 below asserts that $\sigma_{\mathbf{y}}$ has the correct distribution when $x_1 - \gamma_1 \sim N^{-\frac{2}{3}}$.

We also have the following level repulsion estimates. Similar bounds for the measure $\mu_{\mathbf{y}}$ in the bulk were proved in [8, 25].

Theorem 3.2 (Level repulsion estimates for local measures)**.** *Let $\beta > 0$, let ξ be an arbitrary fixed positive constant and assume that K satisfies* (3.1)*. Then there are constants $C, c > 0$ such that for $\mathbf{y} \in \mathcal{R} = \mathcal{R}_K(\xi)$ and for any $s > 0$ we have*

$$\mathbb{P}^{\sigma_{\mathbf{y}}}[y_{K+1} - x_K \leqslant sK^{-1/3}N^{-2/3}] \leqslant C\left(K^2 s\right)^{\beta+1}, \tag{3.8}$$

$$\mathbb{P}^{\sigma_{\mathbf{y}}}[y_{K+1} - x_K \leqslant sK^{-1/3}N^{-2/3}] \leqslant C\left(N^{C\xi} s\right)^{\beta+1} + e^{-N^c}. \tag{3.9}$$

Note that these two bounds are complementary. The first one gives optimal level repulsion for arbitrary small s, but the constant K^2 is not optimal. The second bound improves this constant but at the expenses of an exponentially small additive error.

We remark that statements similar to (3.9) hold for any gap $x_{i+1} - x_i$, not only for the last one with $i = K$. The proofs are very similar, after conditioning on the points $x_{i+1}, x_{i+2}, \ldots, x_K$ being close to their classical locations.

We will prove the rigidity and level repulsion results only for $\sigma_{\mathbf{y}}$ since these bounds are needed in the proof of the main theorems. The proof of the level repulsion bounds for $\sigma_{\mathbf{y}}$, however, verbatim applies to

$\mu_{\mathbf{y}}$. For the rigidity bound, from Theorem 3.1 there exists a set $\mathcal{Y}$ of almost full μ-measure such that for any $\mathbf{y} \in \mathcal{Y}$ we have

$$\mathbb{P}^{\mu_{\mathbf{y}}}\left(|x_k - \gamma_k| > N^{-\frac{2}{3}+\xi+\varepsilon} k^{-\frac{1}{3}}\right) \leqslant e^{-N^c}.$$

The role of the confinement in the definition of σ is to prevent the first particle x_1 to be very negative, since it would destroy the good convexity bound on the Hessian. The reason we have to introduce σ and $\sigma_{\mathbf{y}}$ is that in a technical step (establishing rigidity for the interpolation between local equilibrium measures with two different boundary conditions, see Section 8) we need a superexponential decaying tail probability of the rigidity estimate. We establish such bound only for the confined measure $\sigma_{\mathbf{y}}$ and not for $\mu_{\mathbf{y}}$.

Our main technical result, Theorem 3.3 below, asserts that, for K in a restricted range, the local gap statistics is essentially independent of V and $\mathbf{y}$ for good boundary conditions $\mathbf{y}$ (see (3.4)). For a fixed $\mathbf{y} \in \mathcal{R}$, we define the classical locations $\alpha_j = \alpha_j(\mathbf{y})$ of x_j by the formula

$$\int_0^{\alpha_j} \varrho(s)\mathrm{d}s = \frac{j}{K+1}\int_0^{y_+} \varrho(s)\mathrm{d}s, \qquad j \in [\![1, K]\!], \tag{3.10}$$

i.e., α_j's are the j-th $(K+1)$-quantiles of the density in $J_{\mathbf{y}}$. Recall that the support of ϱ starts from $A = 0$ even though the configuration interval starts from minus infinity.

The core universality result on the local measures is the following theorem. It compares two local measures with potentials V and $\widetilde{V}$ and external configurations $\mathbf{y}$ and $\widetilde{\mathbf{y}}$. For notational simplicity, we will use tilde to refer to objects related to the measure $\widetilde{\mu} := \mu_{\widetilde{V}}$.

Theorem 3.3 (Edge universality for local measures)**.** *Let $\beta \geqslant 1$ and V, $\widetilde{V}$ be $\mathscr{C}^4$ be regular and satisfy (2.3) and (2.4). Assume that the equilibrium density ϱ_V and $\varrho_{\widetilde{V}}$ are supported on a single interval and satisfy (2.8). Fix small positive parameters $\xi, \delta > 0$ and a parameter $0 < \zeta < 1$ that satisfy*

$$C_0\xi < \delta(1-\zeta), \tag{3.11}$$

with a sufficiently large universal constant C_0, and assume that

$$N^{\delta} \leqslant K \leqslant N^{2/5-\delta}. \tag{3.12}$$

Then there is a small $\chi > 0$, independent of N, K, with the following property. Let $\mathbf{y} \in \mathcal{R}^{\#}_{K,V,\beta}(\xi) \cap \mathcal{R}^{*}_{K,V,\beta}(\xi)$ and $\widetilde{\mathbf{y}} \in \mathcal{R}^{\#}_{K,\widetilde{V},\beta}(\xi) \cap \mathcal{R}^{*}_{K,\widetilde{V},\beta}(\xi)$ be two different boundary conditions. Fix $m \in \mathbb{N}$. Then for any $\Lambda \subset [\![1, K^{\zeta}]\!]$, $|\Lambda| = m$, and any smooth, compactly supported observable $O : \mathbb{R}^m \to \mathbb{R}$, we have for N large enough

$$\left|\mathbb{E}^{\sigma_{\mathbf{y}}} O\left(\left(N^{2/3} j^{1/3}(x_j - \alpha_j)\right)_{j\in\Lambda}\right) - \mathbb{E}^{\widetilde{\sigma}_{\widetilde{\mathbf{y}}}} O\left(\left(N^{2/3} j^{1/3}(x_j - \widetilde{\alpha}_j)\right)_{j\in\Lambda}\right)\right| \leqslant N^{-\chi}. \tag{3.13}$$

We remark that, thanks to the conditions (2.8) and (3.12), the points α_j and $\widetilde{\alpha}_j$ (defined by (3.10) with ϱ and $\widetilde{\varrho}$) in (3.13) can both be replaced by γ_j. To see this, we claim that for any $\mathbf{y} \in \mathcal{R}_K$ we have

$$|\alpha_j - \gamma_j| \leqslant C\frac{jN^{-1+\xi}}{K\gamma_j^{1/2}} \leqslant CN^{-2/3+\xi}\,\frac{j^{2/3}}{K}, \tag{3.14}$$

and these estimates are more accurate than the precision detected by the smooth observable O in (3.13) for any $j \leqslant K^{\zeta}$. To prove (3.14), we recall $\gamma_K \sim N^{-2/3}K^{2/3}$ and for $\mathbf{y} \in \mathcal{R}$, we have $|y_{K+1} - \gamma_{K+1}| \leqslant$

$N^{-2/3+\xi}K^{-1/3}$. Since the density has a square root singularity near $A = 0$ (2.5), by assumption $K \geqslant N^{\delta} \gg N^{\xi}$ we have for $\mathbf{y} \in \mathcal{R}$ that

$$\Big| \int_0^{y_+} \varrho(s)\mathrm{d}s - \int_0^{\gamma_{K+1}} \varrho(s)\mathrm{d}s \Big| \leqslant C\gamma_{K+1}^{1/2} N^{-2/3+\xi}K^{-1/3} \leqslant CN^{-1+\xi}.$$

Therefore, for $\mathbf{y} \in \mathcal{R}$ we obtain that

$$\int_0^{\alpha_j} \varrho(s)\mathrm{d}s = \frac{j}{K+1}\Big[\frac{K+1}{N} + \mathrm{O}(N^{-1+\xi})\Big] = \int_0^{\gamma_j} \varrho(s)\mathrm{d}s + \frac{j}{K}\,\mathrm{O}(N^{-1+\xi}).$$

This implies (3.14).

As a consequence of the proof of Theorem 3.3, we also have the following correlation decay estimate.

Theorem 3.4 (Correlation decay near the edge)**.** *Let $\beta \geqslant 1$, V be $\mathscr{C}^4$, regular, and satisfy (2.3), (2.4). Assume that ϱ_V satisfies (2.8). Fix small positive parameters $\xi, \delta > 0$ and assume (3.11), (3.12). Consider the local measure $\sigma_{\mathbf{y}}$ with $\mathbf{y} \in \mathcal{R}^{\#}_{K,V,\beta}(\xi) \cap \mathcal{R}^{*}_{K,V,\beta}(\xi)$. Then there is a constant C, independent of N, K, such that for any two differentiable functions f, q on $J_{\mathbf{y}}$ and large enough N, we have*

$$\langle q(x_i); f(x_j)\rangle_{\sigma_{\mathbf{y}}} \leqslant \frac{N^{C\xi}}{N^{4/3}j^{4/9}}\|q'\|_\infty \|f'\|_\infty, \qquad i \leqslant j \leqslant K, \tag{3.15}$$

where $\langle f; g\rangle_\omega := \mathbb{E}^\omega fg - \mathbb{E}^\omega f\,\mathbb{E}^\omega g$ denotes the covariance. In particular,

$$\Big\langle N^{2/3}i^{1/3}(x_i-\gamma_i); N^{2/3}j^{1/3}(x_j-\gamma_j)\Big\rangle_{\sigma_{\mathbf{y}}} \leqslant \frac{N^{C\xi}i^{1/3}}{j^{1/9}}, \qquad i \leqslant j \leqslant K. \tag{3.16}$$

We remark that the rigidity estimate (3.7) shows that $N^{2/3}i^{1/3}(x_i - i^{2/3}) \leqslant N^{C\xi}$ with a very high probability. Therefore, as long as $i \ll j^{1/3}$, (3.16) is stronger than the trivial bound

$$\big\langle N^{2/3}i^{1/3}(x_i-\gamma_i); N^{2/3}j^{1/3}(x_j-\gamma_j)\big\rangle_{\sigma_{\mathbf{y}}} \leqslant N^{C\xi}$$

obtained from the rigidity estimate. We believe that the optimal estimate on the correlation decay is of the following form:

$$\big\langle N^{2/3}i^{1/3}(x_i-\gamma_i); N^{2/3}j^{1/3}(x_j-\gamma_j)\big\rangle_{\sigma_{\mathbf{y}}} \lesssim \left(\frac{i}{j}\right)^{1/3}, \qquad i \leqslant j \leqslant K, \tag{3.17}$$

and the same decay rate holds for the global measures σ and μ. A heuristic argument that this is the optimal decay rate, at least w.r.t. the GUE measure, will be given in Appendix E. It is based on an extension of the argument in [34]. We note that this decay is quite different from the logarithmic correlation decay in the bulk

$$\big\langle N(x_i-\gamma_i); N(x_j-\gamma_j)\big\rangle_\mu \sim \log\frac{N}{|i-j|+1}$$

which is proven for the GUE measure μ in [34] and conjectured to hold for other ensembles as well.

Theorem 3.3 is our key result. In Sections 4 and 5 we will show how to use Theorem 3.3 to prove the main Theorems 2.7 and 2.1. The proofs of these two theorems follow the arguments used in [25]. The proofs of the auxiliary Theorem 3.1 will be given in Subsection 6.5, and Theorem 3.2 in Appendix D. The proof of Theorem 3.3 will start from Section 7 and will continue until the end of the paper.

4 Edge universality of beta ensembles: proof of Theorem 2.1

In this section, we shall use the edge universality Theorem 3.3 to prove global edge universality Theorem 2.1. Recall the definition of the measure σ with normalization factor Z_σ. We start with he following lemma on properties of σ, defined by (3.2).

Lemma 4.1. *For any bounded observable O we have*

$$\Big|\mathbb{E}^\sigma O - \mathbb{E}^\mu O\Big| \leqslant \|O\|_\infty e^{-N^c}. \tag{4.1}$$

In particular, this implies that μ and σ have the same local statistics and σ also satisfies the following rigidity estimate: for any $\xi > 0$ there exists N_0 and $c > 0$ such that for all $N \geqslant N_0$, $k \in [\![1, N]\!]$, we have

$$\mathbb{P}^\sigma\left(|\lambda_k - \gamma_k| > N^{-\frac{2}{3}+\xi}(\hat{k})^{-\frac{1}{3}}\right) \leqslant\ e^{-N^c}. \tag{4.2}$$

Moreover,

$$\mathbb{P}^\sigma(\mathcal{R}_K^\#(\xi) \cap \mathcal{R}_K^*(\xi)) \geqslant 1 - N^{-c'} \tag{4.3}$$

with some positive constant $c' > 0$.

Proof. Clearly, by $\Theta \geqslant 0$, we have the relation $Z^\sigma \leqslant Z$ among the normalization constants for σ and μ. For a lower bound, from the rigidity estimate (2.11) for x_1 we have

$$1 \geqslant \frac{Z^\sigma}{Z} = \int e^{-2\beta\sum_i \Theta\left(N^{\frac{2}{3}-\xi}x_i\right)}\mathrm{d}\mu \geqslant \mathbb{P}^\mu(x_1 > -N^{-\frac{2}{3}+\xi}) \geqslant 1 - e^{-N^c}.$$

For any bounded nonnegative observable O, we have from the rigidity estimate on μ that

$$\mathbb{E}^\mu O - \|O\|_\infty \mathbb{P}^\mu(x_1 \leqslant -N^{-\frac{2}{3}+\xi}) \leqslant \frac{Z}{Z^\sigma}\mathbb{E}^\mu O \mathbb{1}(x_1 > -N^{-\frac{2}{3}+\xi}) \leqslant \mathbb{E}^\sigma O \leqslant (1 - e^{-N^c})^{-1}\mathbb{E}^\mu O.$$

Using this separately for the positive and negative parts of an arbitrary bounded observable, this proves (4.1). From the rigidity estimate (4.2) we have

$$\mathbb{P}^\sigma(\mathcal{R}_{K+1}(\xi/3)) \geqslant 1 - \exp\left(-N^c\right) \tag{4.4}$$

(notice that the index of $\mathcal{R}$ is $K + 1$ instead of K and we use $\xi/3$ instead of ξ for later convenience). Furthermore, for any $\mathbf{y} \in \mathcal{R}_{K+1}(\xi/3)$, the level repulsion estimate w.r.t. $\sigma_\mathbf{y}$ in the form proved in (3.9) implies

$$\mathbb{P}^{\sigma_\mathbf{y}}\left[y_{K+2} - x_{K+1} \leqslant sN^{-2/3}k^{-1/3}\right] \leqslant C\big(N^{7\xi/9}s\big)^{\beta+1} \tag{4.5}$$

for $s \geqslant \exp(-K^\theta)$. Using (4.4), we see that (4.5) also holds with $\sigma_\mathbf{y}$ replaced by σ. Applying this with $s = N^{-\xi} \ll N^{-7\xi/9}$, we have

$$\mathbb{P}^\sigma(\mathcal{R}_K^\#) \geqslant 1 - N^{-c}. \tag{4.6}$$

The estimates (4.1)–(4.6) also hold for the measure $\widehat{\sigma}$ instead of σ with the same proof.

From the rigidity estimate w.r.t. σ, (4.2), we have for any $\varepsilon > 0$ that

$$\mathbb{E}^\sigma \mathbb{P}^{\sigma_\mathbf{y}}\left(|x_j - \gamma_j| \leqslant N^{-\frac{2}{3}+\varepsilon}j^{-\frac{1}{3}}\ :\ \forall j \in I\right) = \mathbb{P}^\sigma\left(|x_j - \gamma_j| \leqslant N^{-\frac{2}{3}+\varepsilon}j^{-\frac{1}{3}}\ :\ \forall j \in I\right) \geqslant 1 - e^{-N^c}. \tag{4.7}$$

By the estimate (3.14) on $\gamma_j - \alpha_j$, (4.7) also holds if α_j is replaced by γ_j. From the rigidity estimate w.r.t. $\widehat{\sigma}$, we have

$$\mathbb{P}^{\widehat{\sigma}}(A) \geqslant 1 - e^{-N^c}, \quad A := \{\mathbf{y} \in \mathcal{R}_K(\xi) : \mathbb{P}^{\widehat{\sigma}_{\mathbf{y}}}(x_1 \geqslant \gamma_1 - N^{-\frac{2}{3}+\xi}) \geqslant 1/2\}.$$

By (4.1) we have and also the parallel version with σ replaced by $\widehat{\sigma}$, we have

$$|\mathbb{P}^{\widehat{\sigma}}(A) - \mathbb{P}^{\sigma}(A)| \leqslant e^{-N^c}.$$

This guarantees that the second constraint in the definition of $\mathcal{R}^*$ from (3.5) is satisfied for a set of $\mathbf{y}$'s with a high σ-probability. The first constraint is easily satisfied for a large set of $\mathbf{y}$'s by the rigidity w.r.t. σ. Thus we obtain

$$\mathbb{P}^{\sigma}(\mathcal{R}_K^*) \geqslant 1 - e^{-N^c}. \tag{4.8}$$

Combining (4.6) and (4.8), we obtain (4.3). □

Proof of Theorem 2.1. Fix a configuration $\widetilde{\mathbf{y}} \in \widetilde{\mathcal{R}}_K^{\#} \cap \widetilde{\mathcal{R}}_K^*$ where $\widetilde{\mathcal{R}}_K := \mathcal{R}_{K,\tilde{V},\beta}$ and with similar notations for the other sets. Thus we can take expectation of (3.13) with respect to σ and use (4.3) to have

$$\left| \mathbb{E}^{\sigma} \mathbb{1}_{\mathbf{y} \in \mathcal{R}_K^{\#} \cap \mathcal{R}_K^*} \mathbb{E}^{\sigma_{\mathbf{y}}} O\left(\left(N^{2/3} j^{1/3} (x_j - \alpha_j(\mathbf{y})) \right)_{j \in \Lambda} \right) - \mathbb{E}^{\widetilde{\sigma}_{\widetilde{\mathbf{y}}}} O\left(\left(N^{2/3} j^{1/3} (x_j - \widetilde{\alpha}_j(\widetilde{\mathbf{y}})) \right)_{j \in \Lambda} \right) \right| \leqslant N^{-\chi},$$

where we have explicitly indicated the dependence of α_j on $\mathbf{y}$. From (3.14) and $j \leqslant K^{\zeta}$ we have

$$N^{2/3} j^{1/3} |\alpha_j(\mathbf{y}) - \gamma_j| \leqslant N^{\xi} j K^{-1} \leqslant N^{-\chi}$$

provided that

$$N^{\xi+\chi} K^{\zeta-1} \leqslant 1.$$

This condition is guaranteed by the condition (3.11) if $\chi > 0$ is chosen sufficiently small. Under this condition, we have thus proved that

$$\left| \mathbb{E}^{\sigma} \mathbb{1}_{\mathbf{y} \in \mathcal{R}_K^{\#} \cap \mathcal{R}_K^*} \mathbb{E}^{\sigma_{\mathbf{y}}} O\left(\left(N^{2/3} j^{1/3} (x_j - \gamma_j) \right)_{j \in \Lambda} \right) - \mathbb{E}^{\widetilde{\sigma}_{\widetilde{\mathbf{y}}}} O\left(\left(N^{2/3} j^{1/3} (x_j - \widetilde{\alpha}_j) \right)_{j \in \Lambda} \right) \right| \leqslant N^{-\chi}.$$

Recall that the σ-probability of the complement of the set $\mathcal{R}_K^{\#} \cap \mathcal{R}_K^*$ is small, see (4.3), and we can choose $\chi < c'$ where c' is the constant in (4.3). Together with the fact that O is bounded, we can drop the characteristic function $\mathbb{1}_{\mathbf{y} \in \mathcal{R}_K^{\#} \cap \mathcal{R}_K^*}$ at a negligible error and we have

$$\left| \mathbb{E}^{\sigma} O\left(\left(N^{2/3} j^{1/3} (x_j - \gamma_j) \right)_{j \in \Lambda} \right) - \mathbb{E}^{\widetilde{\sigma}_{\widetilde{\mathbf{y}}}} O\left(\left(N^{2/3} j^{1/3} (x_j - \widetilde{\alpha}_j(\widetilde{\mathbf{y}})) \right)_{j \in \Lambda} \right) \right| \leqslant N^{-\chi}.$$

We can now repeat the same argument for the tilde variables. Taking expectation over $\widetilde{\mathbf{y}}$ with respect to $\widetilde{\sigma}$, we see that $\mathbb{E}^{\widetilde{\sigma}_{\widetilde{\mathbf{y}}}}$ can be replaced with $\mathbb{E}^{\widetilde{\sigma}}$ with a negligible error. Finally, using (4.1) we can replace σ with μ and $\widetilde{\sigma}$ with $\widetilde{\mu}$. This proves the global edge universality Theorem 2.1. □

5 Edge universality of Wigner matrices: proof of Theorem 2.7

We will first prove Theorem 2.7 under the assumption that the matrix elements of the normalized matrix satisfy a uniform subexponential decay (2.13). This will be done in the following two steps. First we show that edge universality holds for Wigner matrices with a small Gaussian component. This argument is based upon the analysis of the Dyson Brownian Motion (DBM). In the second step we remove the small Gaussian component by a moment matching perturbation argument.

5.1 Edge universality with a small Gaussian component

We first recall the notion of Dyson's Brownian motion. It describes the evolution of the eigenvalues of a flow of Wigner matrices, $H = H_t$, if each matrix element h_{ij} evolves according to independent (up to symmetry restriction) Ornstein-Uhlenbeck processes. In the Hermitian case, this process for the rescaled matrix elements $v_{ij} := N^{1/2} h_{ij}$ is given by the stochastic differential equation

$$\mathrm{d}v_{ij} = \mathrm{dB}_{ij} - \frac{1}{2} v_{ij} \mathrm{d}t, \qquad i,j \in [\![1, N]\!]$$

where B_{ij}, $i < j$, are independent complex Brownian motions with variance one and B_{ii} are real Brownian motions of the same variance. The real symmetric case is analogous, just β_{ij} are real Brownian motions.

Denote the distribution of the eigenvalues $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_N)$ of $H_t + 2$ at time t by $f_t(\boldsymbol{\lambda})\mu(\mathrm{d}\boldsymbol{\lambda})$ where the Gaussian measure μ is given by (2.2) with $V(x) = \frac{1}{2}(x-2)^2$. (This simple shift ensures that the convention $A = 0$ made at the beginning of Section 3 holds.) The density $f_t = f_{t,N}$ satisfies the forward equation

$$\partial_t f_t = \mathcal{L} f_t, \tag{5.1}$$

where

$$\mathcal{L} = \mathcal{L}_N := \sum_{i=1}^N \frac{1}{2N} \partial_i^2 + \sum_{i=1}^N \left(-\frac{\beta}{4}\lambda_i + \frac{\beta}{2N} \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \right) \partial_i, \quad \partial_i = \frac{\partial}{\partial \lambda_i}, \tag{5.2}$$

with $\beta = 1$ for the real symmetric case and $\beta = 2$ in the complex hermitian case. The initial data f_0 given by the original generalized Wigner matrix. The main result of this section is that edge universality holds for the measure $f_t \mu$ if t is at least a small negative power of N.

Note that, in this section, we always consider the cases $\beta = 1$ or 2, although the proof of the following theorem could be adapted to general $\beta \geqslant 1$.

Theorem 5.1. *Let μ be the Gaussian beta ensemble,* (2.2)*, with quadratic V, and f_t be the solution of* (5.1) *with initial data f_0 given by the original generalized Wigner matrix. Fix an integer $m > 0$ and $\kappa < 1/4$. Then there are positive constants $\mathfrak{b}$ and χ such that for any $t \geqslant N^{-\mathfrak{b}}$ and for any compactly supported smooth observable O we have*

$$\left| \left[\mathbb{E}^{f_t \mu} - \mathbb{E}^{\mu} \right] O\Big(N^{2/3} {p_1}^{1/3} (x_{p_1} - \gamma_{p_1}), \dots, N^{2/3} p_m^{1/3} (x_{p_m} - \gamma_{p_m}) \Big) \right| \leqslant C N^{-\chi},$$

for any $p_1, \dots, p_m \leqslant N^{\kappa}$.

For any $\tau > 0$ define an auxiliary potential $W = W^{\tau}$ by

$$W^{\tau}(\boldsymbol{\lambda}) := \sum_{j=1}^N W_j^{\tau}(\lambda_j), \qquad W_j^{\tau}(\lambda) := \frac{1}{2\tau}(\lambda_j - \gamma_j)^2.$$

The parameter $\tau > 0$ will be chosen as $\tau \sim N^{-\mathfrak{a}}$ where $\mathfrak{a}$ is some positive exponent with $\mathfrak{a} < \mathfrak{b}$.

Definition 5.2. *We define the probability measure* $\mathrm{d}\mu^\tau := Z_\tau^{-1} e^{-N\beta\mathcal{H}^\tau}$*, where the total Hamiltonian is given by*

$$\mathcal{H}^\tau := \mathcal{H} + W^\tau.$$

Here $\mathcal{H}$ *is the Gaussian Hamiltonian given by* (2.2) *with* $V(x) = x^2/2$ *and* $Z_\tau = Z_{\mu^\tau}$ *is the partition function. The measure* μ^τ *will be referred to as the relaxation measure.*

Denote by Q the following quantity

$$Q := \sup_{t\geqslant 0} \frac{1}{N} \int \sum_{j=1}^N (\lambda_j - \gamma_j)^2 f_t(\boldsymbol{\lambda})\mu(\mathrm{d}\boldsymbol{\lambda}).$$

Since H_t is a generalized Wigner matrix for all t, the following rigidity estimate (Theorem 2.2 [30] and Theorem 7.6 [19]) holds:

$$\mathbb{P}^{f_t\mu}\left(|\lambda_k - \gamma_k| > N^{-\frac{2}{3}+\delta\xi}(\hat{k})^{-\frac{1}{3}}\right) \leqslant e^{-N^c}. \tag{5.3}$$

where γ_k is computed w.r.t. the semicircle law and we have used $\delta\xi$ as the small positive exponent needed in the rigidity estimate [19] so that $N^{\delta\xi} \leqslant K^\xi$. Together with a trivial tail estimate from (2.13),

$$\mathbb{P}^{f_t\mu}(|\lambda_i| \geqslant s) \leqslant N^2 \mathbb{P}^{f_t\mu}(|h_{ij}(t)| \geqslant s) \leqslant N^2 \exp(-(s/\sqrt{N})^c), \qquad s > 0, \tag{5.4}$$

this implies that

$$Q \leqslant N^{-2+2\nu}$$

for any $\nu > 0$ if $N \geqslant N_0(\nu)$ is large enough.

Recall the definition of the Dirichlet form w.r.t. a probability measure ω

$$D^\omega(\sqrt{g}) := \sum_{i=1}^N D_i^\omega(\sqrt{g}), \qquad D_i^\omega(\sqrt{g}) := \frac{1}{2N}\int |\partial_i\sqrt{g}|^2 \mathrm{d}\omega = \frac{1}{8N}\int |\partial_i \log g|^2 g \mathrm{d}\omega, \tag{5.5}$$

and the definition of the relative entropy of two probability measures $g\omega$ and ω

$$S(g\omega|\omega) := \int g \log g \mathrm{d}\omega.$$

The $1/N$ prefactor in the definition of the Dirichlet form as well as in (5.2) originates from the $N^{-1/2}$-rescaling of the matrix elements $h_{ij} = N^{-1/2}v_{ij}$.

By the Bakry-Émery criterion [4]), the local relaxation measure satisfies the logarithmic Sobolev inequality, i.e.,

$$S(f\mu^\tau|\mu^\tau) \leqslant C\tau^{-1} D^{\mu^\tau}(\sqrt{f})$$

for any probability measure $f\mu^\tau$.

Now we recall Theorem 2.5 from [27] (the equation (2.37) in [27] has a typo and the correct form should be $S(f_\tau\mu|\omega) \leqslant CN^m$). This theorem was first proved in [29]; a closely related result was obtained earlier in in [23].

Lemma 5.3. *Let* $0 < \tau \leqslant 1$ *be a (possibly* N*-dependent) parameter. Consider the local relaxation measure* μ^τ*. Set* $\psi := \frac{d\mu^\tau}{d\mu}$ *and let* $g_t := f_t/\psi$*. Suppose there is a constant* m *such that*

$$S(f_\tau\mu|\mu^\tau) \leqslant CN^m. \tag{5.6}$$

Fix an $\varepsilon' > 0$. Then for any $t \geqslant \tau N^{\varepsilon'}$ the entropy and the Dirichlet form satisfy the estimates:

$$S(g_t\mu^\tau|\mu^\tau) \leqslant CN^2 Q\tau^{-1}, \qquad D^{\mu^\tau}(\sqrt{g_t}) \leqslant CN^2 Q\tau^{-2}, \tag{5.7}$$

where the constants depend on ε' and m.

We remark that the condition (5.6) is trivially satisfied in our applications for any $\tau \geqslant N^{-2/3+\xi}$ since

$$S(f_\tau\mu|\mu^\tau) \leqslant S(f_\tau\mu|\mu) + \log(Z_\tau/Z) + N\int W^\tau(\boldsymbol{\lambda})f_\tau(\boldsymbol{\lambda})\mathrm{d}\mu(\boldsymbol{\lambda}) \tag{5.8}$$

and $S(f_\tau\mu|\mu) \leqslant S(H_\tau|H_\infty) = N^2 S((h_\tau)_{ij}|(h_\infty)_{ij}) \leqslant CN^m$, where H_∞ is the GOE/GUE matrix. The other two terms in (5.8) satisfy a similar bound by (5.3).

Recall the probability measure σ (3.2) and define q_t by

$$q_t\sigma = f_t\mu = g_t\mu_\tau.$$

From (5.3)–(5.4) and (5.7) (and recalling that we have shifted the eigenvalues in such a way that the left spectral edge -2 is now shifted to 0), we can check that

$$D^\sigma(\sqrt{q_t}) \leqslant 2D^{\mu^\tau}(\sqrt{g_t}) + CN^{4/3}\sum_j \mathbb{E}^{f_t\mu}|\nabla\Theta(N^{2/3-\xi}x_j)|^2 \leqslant 2N^2Q\tau^{-2} + e^{-N^c} \tag{5.9}$$

for any $t \geqslant \tau N^{\varepsilon'}$.

Recall that $\sigma_{\mathbf{y}}$ denotes the conditional measure of σ given $\mathbf{y}$ and $\mathcal{H}^\sigma_{\mathbf{y}}$ its Hamiltonian (3.3). The Hessian of $\mathcal{H}^\sigma_{\mathbf{y}}$ satisfies for all $\mathbf{y} \in \mathcal{R}_K$ and all $\mathbf{u} \in \mathbb{R}^K$ that

$$\langle \mathbf{u}, (\mathcal{H}^\sigma_{\mathbf{y}})''\mathbf{u}\rangle \geqslant \left[N^{4/3-2\xi}\sum_{j\in I}\mathbb{1}(x_j \leqslant -N^{-2/3+\xi})u_j^2 + \sum_{j\in I}V''(x_j)u_j^2 + \frac{1}{N}\sum_{j\in I, k\in I^c}\frac{u_j^2}{(x_j-y_k)^2}\right] \geqslant cN^{1/3}K^{-1/3}\sum_{j\in I}u_j^2. \tag{5.10}$$

In this estimate we used that V'' is bounded from below, see (2.3), and that

$$\frac{1}{N}\sum_{k\in I^c}\frac{1}{(x-y_k)^2} \sim \frac{1}{N}\sum_{k\geqslant K+1}\frac{1}{(N^{\xi-2/3}+N^{-2/3}k^{2/3})^2} \geqslant cN^{1/3}K^{-1/3}$$

holds for any $x \geqslant -N^{-2/3+\xi}$ and $\mathbf{y} \in \mathcal{R}_K$.

Define $q_{t,\mathbf{y}}$ to be the conditional density of $f_t\mu = q_t\sigma$ w.r.t. $\sigma_{\mathbf{y}}$ given $\mathbf{y}$, i.e., it is defined by the relation $q_{t,\mathbf{y}}\sigma_{\mathbf{y}} = (f_t\mu)_{\mathbf{y}}$. From the bound (5.10) we have the logarithmic Sobolev inequality

$$S(q_{t,\mathbf{y}}\sigma_{\mathbf{y}}|\sigma_{\mathbf{y}}) \leqslant C\frac{K^{1/3}}{N^{1/3}}\sum_{i\in I}D_i^{\sigma_{\mathbf{y}}}(\sqrt{q_{t,\mathbf{y}}}). \tag{5.11}$$

Combining it with the entropy inequality, we have

$$\int \mathrm{d}\sigma_{\mathbf{y}}|q_{t,\mathbf{y}}-1| \leqslant C\sqrt{S(q_{t,\mathbf{y}}\sigma_{\mathbf{y}}|\sigma_{\mathbf{y}})} \leqslant C\sqrt{\frac{K^{1/3}}{N^{1/3}}\sum_{i\in I}D_i^{\sigma_{\mathbf{y}}}(\sqrt{q_{t,\mathbf{y}}})}. \tag{5.12}$$

The following Lemma controls the Dirichlet forms $D_i^{\sigma_{\mathbf{y}}}$ for most external configurations $\mathbf{y}$.

Lemma 5.4. *Fix $0 < \mathfrak{a} \leqslant 1$, $\xi, \nu > 0$, and $\tau \geqslant N^{-\mathfrak{a}}$. Suppose the initial data f_0 of the DBM is given by a generalized Wigner ensemble. Then, for any $\varepsilon, \varepsilon' > 0$ and $t \geqslant \tau N^{\varepsilon'}$ there exists a set $\mathcal{G}_{K,t} \subset \mathcal{R}_K(\xi)$ of good boundary conditions $\mathbf{y}$ with*

$$\mathbb{P}^{f_t\mu}(\mathcal{G}_{K,t}) \geqslant 1 - CN^{-\varepsilon} \tag{5.13}$$

such that for any $\mathbf{y} \in \mathcal{G}_{K,t}$ we have

$$\sum_{i \in I} D_i^{\sigma_{\mathbf{y}}}(\sqrt{q_{t,\mathbf{y}}}) \leqslant CN^{3\varepsilon+2\mathfrak{a}+2\nu}. \tag{5.14}$$

Furthermore, for any bounded observable O, we have

$$\big|[\mathbb{E}^{q_{t,\mathbf{y}}\sigma_{\mathbf{y}}} - \mathbb{E}^{\sigma_{\mathbf{y}}}]O(\mathbf{x})\big| \leqslant CK^{1/6}N^{2\varepsilon+\mathfrak{a}+\nu-1/6}. \tag{5.15}$$

We also have

$$\mathbb{E}^{q_{t,\mathbf{y}}\sigma_{\mathbf{y}}}|x_k - \gamma_k| \leqslant CN^{-2/3+\xi}k^{-1/3}, \qquad k \in I. \tag{5.16}$$

The same bounds hold if $\sigma_{\mathbf{y}}$ and $q_{t,\mathbf{y}}$ are replaced with $\widehat{\sigma}_{\mathbf{y}}$ and $\widehat{q}_{t,\mathbf{y}}$ where $\widehat{q}_t$ is defined by $\widehat{q}_t\widehat{\sigma} = f_t\mu$.

Proof. In this proof, we omit the subscript t, i.e., we use $f = f_t$, etc. By definition of the conditional measure and by (5.7) and (5.9), we have for any $\nu > 0$ that

$$\mathbb{E}^{q\sigma}\sum_{i \in I} D_i^{\sigma_{\mathbf{y}}}(\sqrt{q_{t,\mathbf{y}}}) = \sum_{i \in I} D_i^{\sigma}(\sqrt{q}) \leqslant N^2Q\tau^{-2} + e^{-N^c} \leqslant CN^{2\mathfrak{a}+2\nu}.$$

Therefore, by the Markov inequality, (5.14) holds for all $\mathbf{y}$ in a set $\mathcal{G}_K^1$ with $\mathbb{P}^{f\mu}(\mathcal{G}_K^1) \geqslant 1 - CN^{-3\varepsilon}$. Recall from the rigidity estimate (2.11) that $\mathbb{P}^{q\sigma}(\mathcal{R}_K^c) = \mathbb{P}^{f\mu}(\mathcal{R}_K^c)$ is exponentially small. Hence we can choose $\mathcal{G}_K^1$ such that $\mathcal{G}_K^1 \subset \mathcal{R}_K$. The estimate (5.15) now follows from (5.14), (5.11) and (5.12).

Similarly, the rigidity bound (5.3) with respect to $f\mu$ can be translated to the measure $f_{\mathbf{y}}\mu_{\mathbf{y}}$ for most $\mathbf{y}$, i.e., there exists a set $\mathcal{G}_K^2 \subset \mathcal{R}_K$ with

$$\mathbb{P}^{q\sigma}(\mathcal{G}_K^2) = \mathbb{P}^{f\mu}(\mathcal{G}_K^2) \geqslant 1 - \exp\big(-N^c\big),$$

such that for any $\mathbf{y} \in \mathcal{G}_K^2$ and for any $k \in I$, we have

$$\mathbb{P}^{q_{\mathbf{y}}\sigma_{\mathbf{y}}}\Big(|x_k - \gamma_k| \geqslant N^{-2/3}K^{\xi}k^{-1/3}\Big) \leqslant \exp\big(-N^c\big). \tag{5.17}$$

In particular, by setting $\mathcal{G}_K := \mathcal{G}_K^1 \cap \mathcal{G}_K^2$ we can conclude (5.16) for any $\mathbf{y} \in \mathcal{G}_K$. This proves the lemma. □

Lemma 5.5. *Fix $0 < \mathfrak{a} < 1/6$, $\xi, \nu > 0$, and $\tau \geqslant N^{-\mathfrak{a}}$. Suppose the initial data f_0 of the DBM is given by a generalized Wigner ensemble. Then, for any $\varepsilon' > 0$, $t \geqslant \tau N^{\varepsilon'}$, $k \in I$ and $\mathbf{y} \in \mathcal{G}_{K,t}$ (defined in Lemma 5.4), we have*

$$\big|\mathbb{E}^{\sigma_{\mathbf{y}}}x_k - \gamma_k\big| \leqslant N^{-2/3}k^{-1/3}K^{\xi}, \qquad k \in I, \tag{5.18}$$

provided that

$$K^{1/3}N^{-5/6+\nu+\mathfrak{a}+2\varepsilon'} \leqslant N^{-2/3}K^{-1/3+\xi}. \tag{5.19}$$

Notice that we need $\mathfrak{a} < 1/6$ in order that (5.19) has a solution with $K \to \infty$. In our application we will choose $\mathfrak{a}$ arbitrarily close to 0, then we can take any K with $K \leqslant N^{1/4-\delta}$ and still find sufficiently small positive exponents $\nu, \mathfrak{a}, \varepsilon'$ with $\mathfrak{a} + \varepsilon' \leqslant \mathfrak{b}$ so that (5.19) holds. We will not trace the precise interrelation among these exponents. This explains the restriction $\kappa < 1/4$ in Theorem 2.7.

The following proof is essentially the same as the one for Lemma 5.5 in [25].

Proof. We claim that the estimate (5.18) follows from

$$|\mathbb{E}^{\sigma_{\mathbf{y}}} x_k - \mathbb{E}^{q_{t,\mathbf{y}}\sigma_{\mathbf{y}}} x_k| \leqslant K^{1/3} N^{-5/6+\nu+\mathfrak{a}+2\varepsilon'}. \tag{5.20}$$

To see this, we have

$$\left|\mathbb{E}^{\sigma_{\mathbf{y}}} x_k - \gamma_k\right| \leqslant |\mathbb{E}^{\sigma_{\mathbf{y}}} x_k - \mathbb{E}^{q_{t,\mathbf{y}}\sigma_{\mathbf{y}}} x_j k| + |\mathbb{E}^{q_{t,\mathbf{y}}\sigma_{\mathbf{y}}} x_k - \gamma_k| \leqslant N^{-2/3} k^{-1/3} K^{\xi},$$

where we have used (5.16), (5.20) and (5.19). To prove (5.20), we run the reversible dynamics

$$\partial_s h_s = \mathcal{L}_{\mathbf{y}} h_s$$

starting from initial data $h_0 = q_{t,\mathbf{y}}$, where the generator $\mathcal{L}_{\mathbf{y}}$ is the unique reversible generator with the Dirichlet form $D^{\sigma_{\mathbf{y}}}$, i.e.,

$$-\int f \mathcal{L}_{\mathbf{y}}\, g \,\mathrm{d}\sigma_{\mathbf{y}} = \sum_{i\in I} \frac{1}{2N} \int \nabla_i f \cdot \nabla_i g \,\mathrm{d}\sigma_{\mathbf{y}}.$$

Recall that from the convexity bound (5.10), $\tau_K = K^{1/3}/N^{1/3}$ is an upper bound for the time to equilibrium of this dynamics. After differentiation and integration we get,

$$\left[\mathbb{E}^{q_{t,\mathbf{y}}\sigma_{\mathbf{y}}} - \mathbb{E}^{\sigma_{\mathbf{y}}}\right](x_k - \gamma_k) = \int_0^{K^{\varepsilon'}\tau_K} \mathrm{d}u \frac{1}{2N} \int (\partial_k h_u) \mathrm{d}\sigma_{\mathbf{y}} + O(\exp(-cK^{\varepsilon'})).$$

From the Schwarz inequality with a free parameter R, we can bound the last line by

$$\frac{1}{N} \int_0^{K^{\varepsilon'}\tau_K} \mathrm{d}u \int \Big(R(\partial_k \sqrt{h_u})^2 + h_u R^{-1}\Big) \mathrm{d}\sigma_{\mathbf{y}} + O(\exp(-cK^{\varepsilon'})).$$

Dropping the trivial subexponential error term and using that the time integral of the Dirichlet form is bounded by the initial entropy, we can bound the last line by

$$R S(q_{t,\mathbf{y}}\sigma_{\mathbf{y}}|\sigma_{\mathbf{y}}) + \frac{K^{\varepsilon'}\tau_K}{NR}.$$

Using the logarithmic Sobolev inequality for $\sigma_{\mathbf{y}}$ and optimizing the parameter R, we can bound the last term by

$$\begin{aligned}\left|\mathbb{E}^{\sigma_{\mathbf{y}}} x_k - \mathbb{E}^{q_{t,\mathbf{y}}\sigma_{\mathbf{y}}} x_k\right| &\leqslant \tau_K R \sum_{i\in I} D_i^{\sigma_{\mathbf{y}}}(\sqrt{q_{t,\mathbf{y}}}) + \frac{K^{\varepsilon'}\tau_K}{NR} + O(\exp(-cK^{\varepsilon'})) \\ &\leqslant \frac{K^{\varepsilon'}\tau_K}{\sqrt{N}} \Big(\sum_{i\in I} D_i^{\sigma_{\mathbf{y}}}(\sqrt{q_{t,\mathbf{y}}})\Big)^{1/2} + O(\exp(-cK^{\varepsilon'})).\end{aligned}$$

Combining this bound with (5.14) with the choice $\varepsilon = \varepsilon'$, we obtain (5.20). □

We note that if we applied (5.15) with the special choice $O(\mathbf{x}) = x_k$ to control (5.20), then the error estimate would have been much worse. We stress that (5.18) is not an obvious fact although we know that it holds for $\mathbf{y}$ with high probability w.r.t. the equilibrium measure μ. The key point of (5.18) is that it holds for any $\mathbf{y} \in \mathcal{G}_K$, i.e., for a set of $\mathbf{y}$'s with "high probability" w.r.t $f_t\mu$! We also remark that (5.18) holds only in the sense of expectation of x_k and have not yet established that

$$\mathbb{E}^{\sigma_{\mathbf{y}}}\big|x_k - \gamma_k\big| \leqslant N^{-2/3}k^{-1/3}K^{\xi}, \qquad k \in I.$$

We will finally prove this estimate (Theorem 3.1) but only after we prove the rigidity estimate for $\sigma_{\mathbf{y}}$.

We can now prove the main result of this section.

Proof of Theorem 5.1. We will consider only the case $m = 1$ since the general case is only notationally more involved. From the assumption (5.19) the right hand side of (5.15) is smaller than $K^{-1/2}$. Choosing χ sufficiently small, we thus have

$$\Big|\big[\mathbb{E}^{(f_t\mu)_{\mathbf{y}}} - \mathbb{E}^{\sigma_{\mathbf{y}}}\big]O\Big(N^{2/3}p^{1/3}(x_p - \gamma_p)\Big)\Big| \leqslant CK^{-1/2} \leqslant CN^{-\chi}, \tag{5.21}$$

for all $\mathbf{y} \in \mathcal{G}_K$ and $p \leqslant K^{\zeta}$ with the $f_t\mu$-probability of $\mathcal{G}_K$ satisfying (5.13).

We now apply Theorem 3.3 to the same Gaussian beta ensemble with two different boundary conditions so that

$$\Big|\big[\mathbb{E}^{\sigma_{\mathbf{y}}} - \mathbb{E}^{\sigma_{\tilde{\mathbf{y}}}}\big]O\Big(N^{2/3}p^{1/3}(x_p - \gamma_p)\Big)\Big| \leqslant CN^{-\chi},$$

for all $\mathbf{y}, \widetilde{\mathbf{y}} \in \mathcal{R}_K^{\#} \cap \mathcal{R}_K^{*}$ and $p \leqslant K^{\zeta}$. Since $\mathbb{P}^{\sigma}(\mathcal{R}_K^{\#} \cap \mathcal{R}_K^{*}) \geqslant 1 - N^{-c'}$ (see (4.3)), taking the expectation of $\widetilde{\mathbf{y}}$ w.r.t. σ we have thus proved that

$$\Big|\big[\mathbb{E}^{\sigma_{\mathbf{y}}} - \mathbb{E}^{\sigma}\big]O\Big(N^{2/3}p^{1/3}(x_p - \gamma_p)\Big)\Big| \leqslant CN^{-\chi}.$$

We know from (4.1) that

$$\Big|\big[\mathbb{E}^{\sigma} - \mathbb{E}^{\mu}\big]O\Big(N^{2/3}p^{1/3}(x_p - \gamma_p)\Big)\Big| \leqslant CN^{-\chi}.$$

Together with (5.21), we thus have

$$\Big|\big[\mathbb{E}^{(f_t\mu)_{\mathbf{y}}} - \mathbb{E}^{\mu}\big]O\Big(N^{2/3}p^{1/3}(x_p - \gamma_p)\Big)\Big| \leqslant CN^{-\chi}, \tag{5.22}$$

for all $\mathbf{y} \in \mathcal{G}_K \cap \mathcal{R}_K^{\#} \cap \mathcal{R}_K^{*}$. Once we prove that

$$\mathbb{P}^{f_t\mu}(\mathcal{G}_K \cap \mathcal{R}_K^{\#} \cap \mathcal{R}_K^{*}) \geqslant 1 - N^{-\chi} \tag{5.23}$$

then by averaging (5.22) in $\mathbf{y}$ w.r.t. $f_t\mu$ we have

$$\Big|\big[\mathbb{E}^{f_t\mu} - \mathbb{E}^{\mu}\big]O\Big(N^{2/3}p^{1/3}(x_p - \gamma_{p_1})\Big)\Big| \leqslant CN^{-\chi}.$$

and this proves Lemma 5.1.

Finally, we have to prove (5.23). By (5.13) we have that $\mathbb{P}^{f_t\mu}\mathcal{G}_K \geqslant 1 - N^{-\varepsilon}$. We now prove that similar inequality holds for the set $\mathcal{R}_K^{\#}$ and show that $\mathcal{G}_K \subset \mathcal{R}_K^{*}$. This will conclude (5.23) and complete the proof of Lemma 5.1.

Step 1: We first prove that

$$\mathbb{P}^{f_t\mu}(\mathcal{R}_K^{\#}) \geqslant 1 - N^{-c'}. \tag{5.24}$$

Since $f_t\mu$ represents the probability distribution of a generalized Wigner matrix ensemble, from the rigidity estimate (5.3), we have

$$\mathbb{P}^{f_t\mu}(\mathcal{R}_{K+1}) \geqslant 1 - \exp(-N^c). \tag{5.25}$$

From the level repulsion estimate (3.9) with $k = K+1$, we have for any $\mathbf{y} \in \mathcal{R}_{K+1}$ that

$$\mathbb{P}^{\sigma_{\mathbf{y}}}[y_{K+2} - x_{K+1} \leqslant sN^{-2/3}K^{-1/3}] \leqslant C\left(N^{7\xi'/3}s\right)^{\beta+1}.$$

Applying (5.15) with $O(\mathbf{x}) = \mathbb{1}(y_{K+2} - x_{K+1} \leqslant sN^{-2/3}K^{-1/3})$ and using the condition (5.19), we obtain a similar estimate w.r.t. the measure $(f_t\mu)_{\mathbf{y}}$, i.e.,

$$\mathbb{P}^{(f_t\mu)_{\mathbf{y}}}[y_{K+2} - x_{K+1} \leqslant sN^{-2/3}K^{-1/3}] \leqslant C\left(N^{7\xi'/3}s\right)^{\beta+1} + CK^{1/6}N^{2\varepsilon'+\mathfrak{a}+\nu-1/6}. \tag{5.26}$$

This estimate (5.26) and the bound (5.25) with $K+1$ replaced by K imply (5.24) provided $7\xi'/3 \ll \xi$ and (5.19) is satisfied.

Step 2: We now prove that

$$\mathcal{G}_K \subset \mathcal{R}^*_K.$$

By Lemma 5.5, the inequality $|\mathbb{E}^{\sigma_{\mathbf{y}}}x_k - \gamma_k| \leqslant N^{-\frac{2}{3}+\xi}k^{-\frac{1}{3}}$ holds for all $\mathbf{y} \in \mathcal{G}_K$. This verifies the first defining condition of $\mathcal{R}^*$. To check the other defining condition of $\mathcal{R}^*_K$, we now show that

$$\mathbb{P}^{\widehat{\sigma}_{\mathbf{y}}}\left[\Omega\right] \geqslant 1/2, \quad \Omega := \{x_1 \geqslant \gamma_1 - N^{-\frac{2}{3}+\xi}\} \tag{5.27}$$

holds for $\mathbf{y} \in \mathcal{G}_K$. To prove (5.27), for $\mathbf{y} \in \mathcal{G}_K$ we have from (5.15) (applied to $\widehat{\sigma}_{\mathbf{y}}$) that

$$|\mathbb{P}^{\widehat{\sigma}_{\mathbf{y}}}\Omega - \mathbb{P}^{\widehat{q}_{t,\mathbf{y}}\widehat{\sigma}_{\mathbf{y}}}\Omega| \leqslant CK^{1/6}N^{2\varepsilon'+\varepsilon+\mathfrak{a}-1/6}.$$

Under the assumption (5.19), the right hand side of the last equation vanishes as $N \to \infty$. Thus we have

$$\mathbb{P}^{\widehat{\sigma}_{\mathbf{y}}}\left[\Omega\right] \geqslant \mathbb{P}^{\widehat{q}_{t,\mathbf{y}}\widehat{\sigma}_{\mathbf{y}}}\left[\Omega\right] - 1/4.$$

From (5.17), we have $\mathbb{P}^{\widehat{q}_{t,\mathbf{y}}\widehat{\sigma}_{\mathbf{y}}}\Omega \geqslant 1 - e^{-N^c}$ and thus $\mathbb{P}^{\widehat{\sigma}_{\mathbf{y}}}\left[\Omega\right] \geqslant 1/2$ for $\mathbf{y} \in \mathcal{G}_K$. □

5.2 *Removal of the Gaussian convolution*

The last step to complete the proof of edge universality is to approximate arbitrary Wigner matrices by a Gaussian divisible ensemble. We will need the following result.

Theorem 5.6 (Universality of extreme eigenvalues, Theorem 2.4 of [30])**.** *Suppose that we have two $N \times N$ generalized Wigner matrices, $H^{(v)}$ and $H^{(w)}$, with matrix elements h_{ij} given by the random variables $N^{-1/2}v_{ij}$ and $N^{-1/2}w_{ij}$, respectively, with v_{ij} and w_{ij} satisfying the uniform subexponential decay condition* (2.13)*. Let $\mathbb{P}^{\mathbf{v}}$ and $\mathbb{P}^{\mathbf{w}}$ denote the probability and $\mathbb{E}^{\mathbf{v}}$ and $\mathbb{E}^{\mathbf{w}}$ the expectation with respect to these collections of random variables. Suppose that Assumptions* **(A)** *and* **(B)** *hold for both ensembles. If the first two moments of v_{ij} and w_{ij} are the same, i.e.,*

$$\mathbb{E}^{\mathbf{v}}\bar{v}_{ij}^l v_{ij}^u = \mathbb{E}^{\mathbf{w}}\bar{w}_{ij}^l w_{ij}^u, \qquad 0 \leqslant l+u \leqslant 2,$$

then there is an $\varepsilon > 0$ and $\delta > 0$ depending on ϑ in (2.13) *such that or any real parameter s (may depend on N) we have*

$$\mathbb{P}^{\mathbf{v}}(N^{2/3}(\lambda_N - 2) \leqslant s - N^{-\varepsilon}) - N^{-\delta} \leqslant \mathbb{P}^{\mathbf{w}}(N^{2/3}(\lambda_N - 2) \leqslant s) \leqslant \mathbb{P}^{\mathbf{v}}(N^{2/3}(\lambda_N - 2) \leqslant s + N^{-\varepsilon}) + N^{-\delta}$$

for $N \geqslant N_0$ sufficiently large, where N_0 is independent of s. Analogous result holds for the smallest eigenvalue λ_1 and also for extensions to the joint distributions of any finite number of eigenvalues $\lambda_{N-i_1}, \dots, \lambda_{N-i_k}$ as long as $|i_k| \leqslant N^{\varepsilon}$ (or similar results for the smallest eigenvalues).

Given Theorem 5.6, we can now complete the proof of the edge universality for generalized Wigner matrices with subexponential decay. Recall that H_t is the generalized Wigner matrix whose matrix elements evolve by independent OU processes. In Theorem 5.1 we have proved that the statistics of eigenvalues at the spectral edge of H_t, for $t \geqslant N^{-\mathfrak{a}+\varepsilon'}$ and for *any* initial generalized matrix H, is the same as the standard Gaussian one in the corresponding symmetry class. We now construct an auxiliary Wigner matrix H_0 (see, e.g., Lemma 3.4 of [28] which allows us to match) such that the first two moments of H_t (with $t = N^{-c'}$ for some small $c' > 0$) and the first two moments of the *original* matrix $H^{\mathbf{v}}$ are identical. The edge statistics of $H^{\mathbf{v}}$ and H_t coincide by Theorem 5.6 and the edge statistics of H_t are identical to those of the standard GOE/GUE by Theorem 5.1. This completes our proof of Theorem 2.7. □

6 Rigidity of the particles

Most of this section is devoted to proving Theorem 2.4 which asserts the rigidity of the particles under the measure μ at the optimal scale up to the edge (which, for us, means a control throughout the support of the equilibrium measure including the edge). We recall that the same statement holds for the measure σ (Lemma 4.1).

Our method to prove rigidity is a multiscale analysis, initiated for the bulk particles in [8, 9]. It is a bootstrap argument where concentration and accuracy bounds are proved in tandem, gradually for smaller and smaller scales. Concentration bound means a control on the fluctuation of a particle around its mean; this is obtained by a local logarithmic Sobolev inequality (for non-convex V we need an extra convexification argument). To estimate the log-Sobolev constant we use rigidity on a larger scale. The next step is to identify the mean, this is achieved by the first loop equation, where the error term involves the improved concentration bound. This leads to a better accuracy and thus better rigidity. This information can be used to improve the concentration bound on a smaller scale, etc. In this paper we prove rigidity up to the edge, which involves new difficulties: the loop equation is less stable since the density vanishes near the edge. Moreover, the loop equation is used to improve the accuracy of one specific particle (the leftmost one, λ_1), whose rigidity cannot originate in the pairwise interaction from surrounding particles.

This extra difficulty (lack of a natural boundary on the left) is also critical in the last subsection, where we prove Theorem 3.1, i.e., the rigidity of the particles under the conditional measure $\sigma_{\mathbf{y}}$ with a Gaussian tail. Extra convexity (hence rigidity) on the left of the first particle is the reason for introducing the modification $\sigma_{\mathbf{y}}$ of $\mu_{\mathbf{y}}$ which artificially confines the first particle.

Another extra difficulty consists in improving the accuracy without assuming that V is analytic. This analyticity condition was essential in the works [36, 50] and the previous optimal bulk rigidity estimates [8, 9]. It turns out that the analyticity condition can be replaced by a much weaker smoothness assumption by a more careful analysis of the first loop equation, see (6.18) and (6.38).

In this section we disregard the shift convention which sets $A = 0$.

6.1 Statement of the results

For any fixed N, let the *classical position* $\gamma_k^{(N)}$ of the k-th particle under $\mu^{(N)}$ be defined by

$$\int_{-\infty}^{\gamma_k^{(N)}} \varrho_1^{(N)}(s)\mathrm{d}s = \frac{k}{N}, \tag{6.1}$$

where $\varrho_1^{(N)}$ is the density of $\mu^{(N)}$. Recall that γ_k from (2.6) denotes the limiting classical location.

Definition 6.1. *In the following definitions, the potential V and $\beta > 0$ are fixed.*

(i) We say that **rigidity** *at scale a holds if for any $\varepsilon > 0$, there are constants $c > 0$ and N_0 such that for any $N \geqslant N_0$ and $k \in [\![1, N]\!]$ we have*

$$\mathbb{P}^{\mu}\left(|\lambda_k - \gamma_k| > N^{-\frac{2}{3}+a+\varepsilon}(\hat{k})^{-\frac{1}{3}}\right) \leqslant\ e^{-N^c}.$$

(ii) We say that **concentration** *at scale a holds if for any $\varepsilon > 0$, there are constants $c > 0$ and N_0 such that for any $N \geqslant N_0$ and $k \in [\![1, N]\!]$ we have*

$$\mathbb{P}^{\mu}\left(|\lambda_k - \mathbb{E}^{\mu}(\lambda_k)| > N^{-\frac{2}{3}+a+\varepsilon}(\hat{k})^{-\frac{1}{3}}\right) \leqslant\ e^{-N^c}.$$

(iii) We say that **accuracy** *at scale a holds if for any $\varepsilon > 0$, there is a constant N_0 such that for any $N \geqslant N_0$ and $k \in [\![1, N]\!]$ we have*

$$\left|\gamma_k - \gamma_k^{(N)}\right| \leqslant N^{-\frac{2}{3}+a+\varepsilon}(\hat{k})^{-\frac{1}{3}}.$$

For the proof of Theorem 2.4 the main steps are the concentration and accuracy improvements hereafter, proved in the following subsections.

Proposition 6.2. *Let V be $\mathscr{C}^2$, regular with equilibrium density supported on a single interval $[A, B]$, and satisfy (2.3), (2.4). Then rigidity at scale a implies concentration at scale $a/2$.*

Proposition 6.3. *Let V be $\mathscr{C}^4$, regular with equilibrium density supported on a single interval $[A, B]$, and satisfy (2.3), (2.4). Then rigidity at scale a implies accuracy at scale $11a/12$.*

Remark 6.4. *Notice that the accuracy improves from scale a only to scale $11a/12$ instead of $3a/4$ as it was achieved in the bulk case (see Proposition 3.13 in [8]). This weaker control is due to some difficult estimates near the edge that have not been optimized.*

Proof of Theorem 2.4. It is known that rigidity at scale 1 holds. More precisely, for any $\varepsilon > 0$ there are positive constants c_1, c_2 such that, for all $N \geqslant 1$,

$$\mathbb{P}^{\mu}\left(\exists k \in [\![1, N]\!] \mid |\lambda_k - \gamma_k| \geqslant \varepsilon\right) \leqslant c_1 e^{-c_2 N}. \tag{6.2}$$

For eigenvalues in the bulk, (6.2) follows from the large deviations for the empirical spectral measure with speed N^2, see [2,6]. For the extreme eigenvalues the large deviations principle with speed N is proved in [5] for the GOE case, and extended in [2] Theorem 2.6.6, for the general case (up to a condition on the partition function that follows from Theorem 1 (iii) in [50]).

We now use Propositions 6.2 and 6.3 to obtain that concentration and accuracy hold at scale 11/12. We just need to prove that concentration and accuracy at some scale $b>0$ imply rigidity the same scale b. Then a simple induction on scales shows that rigidity holds on scale $(11/12)^m$ for any integer m, i.e., it holds at any positive scale ξ.

To show the key part of the induction step, assume that concentration and accuracy hold at scale b. Fix any $k\in[\![1,N]\!]$. Then for any $\varepsilon>0$ we have

$$\begin{aligned}\mathbb{E}^{\mu}\#\Big\{\lambda_i\leqslant \mathbb{E}^{\mu}(\lambda_k)-N^{-\frac23+b+\frac{\varepsilon}{2}}(\hat k)^{-\frac13}\Big\} &= \sum_{\ell=1}^{N}\mathbb{P}^{\mu}\Big\{\lambda_\ell<\mathbb{E}^{\mu}(\lambda_k)-N^{-\frac23+b+\frac{\varepsilon}{2}}(\hat k)^{-\frac13}\Big\}\\ &\leqslant k-1+(N-k+1)\mathbb{P}^{\mu}\Big\{\lambda_k<\mathbb{E}^{\mu}(\lambda_k)-N^{-\frac23+b+\frac{\varepsilon}{2}}(\hat k)^{-\frac13}\Big\}\leqslant k\end{aligned}$$

for large enough N, independently of k, since the probability in the last line is subexponentially small by concentration on scale b. As $\gamma_k^{(N)}$ is defined by $\mathbb{E}^{\mu}(\#\{\lambda_i\leqslant\gamma_k^{(N)}\})=k$, this implies that $\gamma_k^{(N)}\geqslant \mathbb{E}^{\mu}(\lambda_k)-N^{-\frac23+b+\frac{\varepsilon}{2}}(\hat k)^{-\frac13}$ for some large enough N, independent of k. In the same way one can get the upper bound, which yields

$$|\gamma_k^{(N)}-\mathbb{E}^{\mu}\lambda_k|\leqslant N^{-\frac23+b+\varepsilon}(\hat k)^{-\frac13}$$

for large enough N. As we have accuracy at scale b, the same conclusion holds when replacing $\gamma_k^{(N)}$ by γ_k. We thus proved, for any $\varepsilon>0$, the existence of some $C>0$ such that for all N and k we have

$$|\gamma_k-\mathbb{E}^{\mu}\lambda_k|\leqslant CN^{-\frac23+b+\frac{\varepsilon}{2}}(\hat k)^{-\frac13}. \tag{6.3}$$

The conclusion now easily follows from

$$\begin{aligned}&\mathbb{P}^{\mu}\Big\{|\lambda_k-\gamma_k|\geqslant N^{-\frac23+b+\varepsilon}(\hat k)^{-\frac13}\Big\}\\ &\qquad\leqslant \mathbb{P}^{\mu}\left\{|\lambda_k-\mathbb{E}^{\mu}\lambda_k|\geqslant \frac12 N^{-\frac23+b+\varepsilon}(\hat k)^{-\frac13}\right\}+\mathbb{1}\left(|\gamma_k-\mathbb{E}^{\mu}\lambda_k|\geqslant\frac12 N^{-\frac23+b+\varepsilon}(\hat k)^{-\frac13}\right).\end{aligned}$$

The first term can be bounded by the concentration hypothesis, the second term is 0 for large enough N, thanks to (6.3). □

6.2 Initial estimates for non-analytic potentials

Let h be a continuous and bounded function. Consider the probability distribution on the simplex $\lambda_1\leqslant\ldots\leqslant\lambda_N$ given by

$$\mu^{(N,h)}(\mathrm{d}\boldsymbol{\lambda})\sim e^{-\beta(N\mathcal{H}(\lambda)+\sum_{k=1}^{N}h(\lambda_k))}\mathrm{d}\boldsymbol{\lambda},$$

where $\mathcal{H}$ is defined in (1.1). We denote by $m_{N,h}$ the Stieltjes transform for the measure $\mu^{(N,h)}$:

$$m_{N,h}(z)=\mathbb{E}^{\mu^{(N,h)}}\left(\frac1N\sum_{k=1}^{N}\frac{1}{z-\lambda_k}\right). \tag{6.4}$$

In the following, it will be useful to have the density supported strictly in a compact interval: for given $\kappa>0$, define the following variant of $\mu^{(N,h)}$ conditioned to have all particles in $[A-\kappa,B+\kappa]$:

$$\mu^{(N,h,\kappa)}(\mathrm{d}\lambda)=\frac{1}{Z_{N,\kappa}}\prod_{1\leqslant i<j\leqslant N}|\lambda_i-\lambda_j|^{\beta}\prod_{k=1}^{N}e^{-\beta\left(\frac N2 V(\lambda_k)+h(\lambda_k)\right)}\mathbb{1}_{\lambda_k\in[A-\kappa,B+\kappa]}\mathrm{d}\lambda_1\ldots\mathrm{d}\lambda_N. \tag{6.5}$$

We will choose κ to be small, fixed number. Let $\varrho_k^{(N,h,\kappa)}$ denote the correlation functions and $m_{N,h,\kappa}(z)$ the Stieltjes transform, defined in the same way as (2.7) and (6.4), but for the underlying measure $\mu^{(N,h,\kappa)}$. Then Lemma 1 in [10] (strictly speaking this result is given in [10] only for $h \equiv 0$, but the proof works for any fixed h) states that under condition (2.4), for some large enough κ there exists some $c > 0$, depending only on V, such that for any $x_1, \dots, x_k \in [A-\kappa, B+\kappa]$, we have

$$\left|\varrho_k^{(N,h,\kappa)}(x_1,\dots,x_k) - \varrho_k^{(N,h)}(x_1,\dots,x_k)\right| \leqslant \varrho_k^{(N,h,\kappa)}(x_1,\dots,x_k)e^{-cN}, \tag{6.6}$$

and for $x_1,\dots,x_j \notin [A-\kappa, B+\kappa]$, $x_{j+1},\dots,x_k \in [A-\kappa,B+\kappa]$,

$$\varrho_k^{(N,h)}(x_1,\dots,x_k) \leqslant e^{-cN\sum_{i=1}^{j}\log|x_i|}. \tag{6.7}$$

The estimates (6.6) and (6.7) actually also hold for arbitrarily small fixed $\kappa > 0$ thanks to the large deviations estimates (6.2), which holds not only for $\mu^{(N)}$ but also for $\mu^{(N,h)}$. From now we fix this small parameter $\kappa > 0$. The following Lemma relates estimates on $m_{N,h} - m$ and concentration of linear statistics of the particles.

Lemma 6.5. *Let V be $\mathscr{C}^4$, regular such that the equilibrium density ϱ_V is supported on a single interval $[A,B]$ and satisfies (2.3), (2.4). Let h_1, h_2 be $\mathscr{C}^2$ functions such that $\|h_1\|_\infty, \|h_1'\|_\infty, \|h_1''\|_\infty < \infty$, and the same for h_2. Let $a \in (0,1/2)$ and $\varepsilon > 0$. Assume that for any $\Im(z) = \eta \in (N^{-1/2}, N^{-a})$ and $s \in (-\beta,\beta)$ we have*

$$\left|m_{N,(1+s)h_1}(z) - m(z)\right| \leqslant C\frac{N^\varepsilon}{|(z-A)(z-B)|^{1/2}\,N\eta^2}. \tag{6.8}$$

Then there is a constant $c > 0$ such that, for any $N \geqslant 1$, we have

$$\mathbb{P}^{\mu^{(N,h_1)}}\left(\left|\sum_{k=1}^N h_2(\lambda_k) - N\int h_2(s)\varrho(s)\mathrm{d}s\right| > N^{2a+2\varepsilon}\right) \leqslant e^{-N^c}. \tag{6.9}$$

Proof. Let $\kappa > 0$ be a small constant and $C > 0$ be chosen such that for any $E \in I_\kappa := [A-\kappa, B+\kappa]$, $\eta \in (0, N^{-a})$, and $s \in (-\beta,\beta)$ we have

$$\left|m_{N,(1+s)h_1}(z) - m(z)\right| \leqslant C\frac{(\log N)^{1/2}}{N^{1/2}\eta}. \tag{6.10}$$

This inequality was proved in [46], Theorem 2.3 (ii) Let χ be a smooth nonnegative cutoff function; $\chi = 1$ on $[0, N^{-a}/2]$, $\chi = 0$ on $[N^{-a},\infty)$, $\|\chi'\|_\infty = \mathrm{O}(N^a)$. Let $\widetilde{h}_2$ be $\mathscr{C}^2$, compactly supported on I_κ, such that $h_2 = \widetilde{h}_2$ on $I_{\kappa/2}$, for some $\kappa > 0$. From the large deviations estimate (6.2) we have, for any $\varepsilon > 0$,

$$\mathbb{P}^{\mu^{(N,h_1)}}\left(\left|\sum_{k=1}^N (h_2(\lambda_k) - \widetilde{h}_2(\lambda_k))\right| > N^\varepsilon\right) \leqslant e^{-N^c}.$$

for some $c > 0$. As a consequence, to prove (6.9), we can assume that h_2 is supported on I_κ. By the Helffer-Sjöstrand formula (see formula (B.13) in [22]),

$$\int h_2(u)(\varrho_1^{(N,(1+s)h_1)}(u) - \varrho(u))\mathrm{d}u = \mathrm{O}\left(\iint_{x\in I_\kappa,\eta>0} (\eta\chi(\eta) + |\chi'(\eta)|)\left|m_{N,(1+s)h_1}(x+\mathrm{i}\eta) - m(x+\mathrm{i}\eta)\right|\mathrm{d}x\mathrm{d}\eta\right).$$

The term involving χ' can be evaluated using (6.8), and is bounded by N^{-1+2a}. For the χ term, we bound $m_{N,(1+s)h_1}(z)-m(z)$ by (6.8) if $\eta \geqslant N^{-1/2}$ and by (6.10) if $\eta \in (0, N^{-1/2})$. We obtain

$$\int h_2(u)(\varrho_1^{(N,(1+s)h_1)}(u)-\varrho(u))\mathrm{d}u = \mathrm{O}\left(\frac{N^{2a}}{N}\right).$$

The remainder of the proof is a classical argument: using the above estimate we get

$$\frac{\mathrm{d}}{\mathrm{d}s}\log \mathbb{E}^{\mu^{(N,h_1)}}\left(e^{s\left(\sum_{k=1}^N h_2(\lambda_k)-N\int h_2(u)\varrho(u)\mathrm{d}u\right)}\right) = \mathbb{E}^{\mu^{(N,(1+s/\beta)h_1)}}\left(\sum_{k=1}^N h_2(\lambda_k) - N\int h_2(u)\varrho(u)\mathrm{d}u\right) = \mathrm{O}(N^{2a}).$$

This yields

$$\mathbb{E}^{\mu^{(N,h_1)}}\left(e^{\sum_{k=1}^N h_2(\lambda_k)-N\int h_2(s)\varrho(s)\mathrm{d}s}\right) + \mathbb{E}^{\mu^{(N,h_1)}}\left(e^{-\left(\sum_{k=1}^N h_2(\lambda_k)-N\int h_2(s)\varrho(s)\mathrm{d}s\right)}\right) \leqslant e^{cN^{2a}},$$

and one concludes by the exponential Markov inequality. □

The following lemma provides almost optimal estimates for $m_{N,h}-m$ for $\eta = \Im(z)$ till order 1. For non-analytic V, it improves previous estimates by Pastur and Shcherbina by a factor $\sqrt{N}$, and relies on their initial estimates proved in [46].

Lemma 6.6. *Let V be $\mathscr{C}^4$, regular such that the equilibrium density ϱ_V is supported on a single interval $[A,B]$ and satisfy (2.3), (2.4). Let h be a $\mathscr{C}^2$ function with $\|h\|_\infty, \|h'\|_\infty, \|h''\|_\infty < \infty$. Then for any $\varepsilon > 0$ there exists a constant $C = C(V,\varepsilon,\|h'\|_\infty)$ such that, for any $E \in [A-\kappa, B+\kappa]$, $\eta \in (0, N^{-\varepsilon})$, we have*

$$|(z-A)(z-B)|^{1/2}\,|m_{N,h}(z)-m(z)| \leqslant C\ \frac{N^{\varepsilon}}{N\eta^2}.$$

Proof. Let $I_\kappa = [A-\kappa, B+\kappa]$ and $d(\xi) = \inf_{s\in I_\kappa}|\xi - s|$. Thanks to the estimates (6.6) and (6.7), we just need to prove the lemma for the Stieltjes transform $m_{N,h,\kappa}$ instead of $m_{N,h}$.

For any $a \in (0,1)$, let $\mathcal{P}(a)$ be the following property: for any $\varepsilon > 0$ there exists a constant $C = C(V,a,\varepsilon,\|h'\|_\infty)$ such that, for any $E \in [A-\kappa, B+\kappa]$, we have

$$|(z-A)(z-B)|^{1/2}\,|m_{N,h,\kappa}(z)-m(z)| \leqslant C\ \frac{N^{\varepsilon}}{N\eta^2} \quad \text{for } \eta \in (0, N^{-a}), \tag{6.11}$$

$$|(z-A)(z-B)|^{1/2}\,|m_{N,h,\kappa}(z)-m(z)| \leqslant C\ \frac{N^{2a+\varepsilon}}{N} \quad \text{for } \eta \in [N^{-a}, 1]. \tag{6.12}$$

We will prove that $\mathcal{P}(a)$ implies $\mathcal{P}(a/2)$, which concludes the proof of the lemma by induction, as $\mathcal{P}(1/2)$ holds: Pastur and Shcherbina (see [46][1] Theorem 2.3 (ii)): proved that

$$m_{N,h,\kappa}(\xi) - m(\xi) = \mathrm{O}\left(\frac{(\log N)^{1/2}}{N^{1/2}d(\xi)}\right),\ \frac{1}{N^2}\operatorname{Var}_{\mu^{(N,h,\kappa)}}\left(\sum_{k=1}^N \frac{1}{\xi-\lambda_k}\right) = \mathrm{O}\left(\frac{\log N}{Nd(\xi)^2}\right), \tag{6.13}$$

the second estimate being useful later along the proof. Here we used that $\eta|(z-A)(z-B)|^{1/2} \leqslant d(z)$.

[1]Strictly speaking these estimates were proved for $h \equiv 0$, but the analysis in [46] extends to our context in a straightforward way, when $\|h\|_\infty, \|h'\|_\infty, \|h''\|_\infty < \infty$.

Assume that $\mathcal{P}(a)$ holds. To prove $\mathcal{P}(a/2)$, we will need the quasi-analytic extension of V of order three:

$$\widetilde{V}(z) = V(E) + \mathrm{i}\eta V'(E) - \frac{\eta^2}{2}V''(E). \tag{6.14}$$

Note that $V = \widetilde{V}$ on the real axis. One easily checks that

$$\partial_{\bar{z}}\partial_E \widetilde{V}(z) = -\frac{\eta^2}{2}V^{(4)}(E). \tag{6.15}$$

The first loop equation and its limit are (see [31,36,50] for various proofs), for any $\xi \notin \mathbb{R}$,

$$m_{N,h}(\xi)^2 + \int_{\mathbb{R}} \frac{V'(s)+N^{-1}h'(s)}{\xi - s}\varrho_1^{(N,h)}(s)\mathrm{d}s = \frac{1}{N}\left(\frac{2}{\beta}-1\right)m'_{N,h}(\xi) + \frac{1}{N^2}\operatorname{Var}_{\mu^{(N,h)}}\left(\sum_{k=1}^N \frac{1}{\xi-\lambda_k}\right), \tag{6.16}$$

$$m(\xi)^2 + \int_{\mathbb{R}} \frac{V'(s)}{\xi - s}\varrho(s)\mathrm{d}s = 0. \tag{6.17}$$

Here $\operatorname{Var}_\mu X := \mathbb{E}^\mu X^2 - (\mathbb{E}^\mu X)^2$, in particular $\operatorname{Var}_\mu X$ may be complex. We choose to write the difference of both equations in the following way:

$$(m_{N,h}(\xi)-m(\xi))^2 + (2m(\xi)-\partial_E\widetilde{V}(\xi))(m_{N,h}(\xi)-m(\xi)) + \int_{\mathbb{R}} \frac{\partial_E\widetilde{V}(\xi)-V'(s)}{\xi - s}(\varrho_1^{(N,h)}(s)-\varrho(s))\mathrm{d}s$$
$$+\frac{1}{N}\int_{\mathbb{R}} \frac{h'(s)}{\xi-s}\varrho_1^{(N,h)}(s)\mathrm{d}s - \frac{1}{N}\left(\frac{2}{\beta}-1\right)m'_{N,h}(\xi) - \frac{1}{N^2}\operatorname{Var}_{\mu^{(N,h)}}\left(\sum_{k=1}^N \frac{1}{\xi-\lambda_k}\right) = 0.$$

Thanks to the estimates (6.6) and (6.7), the above equation also holds when all considered quantities are with respect to the measure $\mu^{(N,h,\kappa)}$ instead of $\mu^{(N,h)}$, up to an exponentially small error term which is uniform in $\{d(\xi) > N^{-10}\}$:

$$(m_{N,h,\kappa}(\xi)-m(\xi))^2 + (2m(\xi)-\partial_E\widetilde{V}(\xi))(m_{N,h,\kappa}(\xi)-m(\xi)) + b_N(\xi) - c_N(\xi) = \mathrm{O}\left(e^{-cN}\right), \tag{6.18}$$

$$b_N(\xi) := \int_{\mathbb{R}} \frac{\partial_E\widetilde{V}(\xi)-V'(s)}{\xi - s}(\varrho_1^{(N,h,\kappa)}(s)-\varrho(s))\mathrm{d}s, \tag{6.19}$$

$$c_N(\xi) := -\frac{1}{N}\int_{\mathbb{R}} \frac{h'(s)}{\xi-s}\varrho_1^{(N,h,\kappa)}(s)\mathrm{d}s + \frac{1}{N}\left(\frac{2}{\beta}-1\right)m'_{N,h,\kappa}(\xi) + \frac{1}{N^2}\operatorname{Var}_{\mu^{(N,h,\kappa)}}\left(\sum_{k=1}^N \frac{1}{\xi-\lambda_k}\right). \tag{6.20}$$

Take z such that $\Im z = \eta \in (N^{-a}, 1)$, let $\delta \in (N^{-a}/4, \eta/2)$ be chosen later, and consider the domain $\Omega_\delta = \{\xi \mid d(\xi) \leqslant \delta\}$, and $\partial\Omega_\delta$ its boundary, encircling I_κ but not z. We also use the notation, for $\xi \notin I_\kappa$,

$$r(\xi) = \frac{((A-\xi)(B-\xi))^{1/2}}{2m(\xi)-\partial_E\widetilde{V}(\xi)},$$

where the branch of the numerator is chosen so that $((A-\xi)(B-\xi))^{1/2} \sim \xi$ as $|\xi| \to \infty$. One can check that r is continuous in $\mathbb{C}$: thanks to the equilibrium equation $m(s) = \frac{1}{2}V'(s)$ $(s \in [A,B])$ the real part of $2m(\xi) - \partial_E\widetilde{V}(\xi)$ vanishes on $[A,B]$, and the imaginary parts of the numerator and the denominator both change signs across $[A,B]$. Moreover, thanks to the square root singularity of ρ at A^- and B^+, the following bounds easily hold:

$$c \leqslant |r(\xi)| \leqslant c^{-1} \tag{6.21}$$

uniformly in Ω_η, for some $c>0$. Multiplying (6.18) by $r(\xi)$ and integrating counterclockwise, one can write

$$\int_{\partial\Omega_\delta} \frac{(m_{N,h,\kappa}(\xi)-m(\xi))((A-\xi)(B-\xi))^{1/2}}{z-\xi}\mathrm{d}\xi = \int_{\partial\Omega_\delta} \frac{-(m_{N,h,\kappa}(\xi)-m(\xi))^2+c_N(\xi)}{z-\xi} r(\xi)\mathrm{d}\xi \tag{6.22}$$

$$-\int_{\partial\Omega_\delta} \frac{b_N(\xi)}{z-\xi} r(\xi)\mathrm{d}\xi + \mathrm{O}(e^{-cN}). \tag{6.23}$$

Since $m_{N,h,\kappa}(\xi)$ and $m(\xi)$ are both Stieltjes transforms of a probability measure, we have $|m_{N,h,\kappa}(\xi)-m(\xi)| = \mathrm{O}(|\xi|^{-2})$, thus $(m_{N,h,\kappa}(\xi)-m(\xi))((A-\xi)(B-\xi))^{1/2} = \mathrm{O}(|\xi|^{-1})$ as $|\xi|\to\infty$. So the left hand side of (6.22) is $2\pi\mathrm{i}((A-z)(B-z))^{1/2}(m_{N,h,\kappa}(z)-m(z))$, by the residue theorem. Moreover, we have the estimates (6.13) and the trivial bounds

$$\frac{1}{N} m'_{N,h,\kappa}(\xi) = \mathrm{O}\left(\frac{1}{Nd(\xi)^2}\right), \qquad \frac{1}{N}\int_{\mathbb{R}} \frac{h'(s)}{\xi-s}\varrho_1^{N,h,\kappa}(s)\mathrm{d}s = \mathrm{O}\left(\frac{1}{Nd(\xi)}\right).$$

Together with (6.13), this implies that the right hand side of (6.22) is $\mathrm{O}\left(\frac{(\log N)^2}{N\delta^2}\right)$.

Finally, to estimate (6.23), we will use the induction hypothesis $\mathcal{P}(a)$. We first introduce the notations (for $t>0$)

$$\Omega_{\delta,t} = \{\omega\in\Omega_\delta \mid \Im(\omega) > t\},\ \Omega_{\delta,-t} = \{\omega\in\Omega_\delta \mid \Im(\omega) < -t\}.$$

By first using the continuity of r and b_N at $\Im(\xi)=0$ and then Green's formula separately in $\Omega_{\delta,t}$, $\Omega_{\delta,-t}$, we obtain (all contour integrals being counterclockwise)

$$\begin{aligned}
\int_{\partial\Omega_\delta} \frac{b_N(\xi)}{z-\xi} r(\xi)\mathrm{d}\xi &= \lim_{t\to 0^+}\left(\int_{\partial\Omega_{\delta,t}} \frac{b_N(\xi)}{z-\xi} r(\xi)\mathrm{d}\xi + \int_{\partial\Omega_{\delta,-t}} \frac{b_N(\xi)}{z-\xi} r(\xi)\mathrm{d}\xi\right)\\
&= \mathrm{O}\left(\iint_{\Omega_\delta\setminus\mathbb{R}} \frac{1}{|z-\xi|}|\partial_{\bar\xi}(b_N(\xi)r(\xi))|\mathrm{d}\xi\mathrm{d}\bar\xi\right)\\
&= \mathrm{O}\left(\iint_{\Omega_\delta\setminus\mathbb{R}} \frac{1}{|z-\xi|}|\partial_{\bar\xi}(b_N(\xi))|\mathrm{d}\xi\mathrm{d}\bar\xi\right) + \mathrm{O}\left(\iint_{\Omega_\delta\setminus\mathbb{R}} \frac{1}{|z-\xi|}|b_N(\xi)\partial_{\bar\xi} r(\xi)|\mathrm{d}\xi\mathrm{d}\bar\xi\right),
\end{aligned} \tag{6.24}$$

where we used (6.21). A straightforward calculation from (6.15) and (6.21) yields

$$\partial_{\bar\xi} b_N(\xi) = -\frac{\Im(\xi)^2}{2}V^{(4)}(E)(m_{N,h,\kappa}(\xi)-m(\xi)), \qquad \partial_{\bar\xi} r(\xi) = \mathrm{O}\left(\frac{\partial_{\bar\xi}(2m(\xi)-\partial_E\widetilde{V}(\xi))}{|(A-\xi)(B-\xi)|^{1/2}}\right) = \mathrm{O}\left(\frac{(\Im(\xi))^2)}{|(A-\xi)(B-\xi)|^{1/2}}\right). \tag{6.25}$$

Moreover, as V is of class $\mathscr{C}^4$, the functions $s\mapsto \Re\left(\frac{\partial_E\widetilde{V}(\xi)-V'(s)}{\xi-s}\right)$, $s\mapsto\Im\left(\frac{\partial_E\widetilde{V}(\xi)-V'(s)}{\xi-s}\right)$ have their first two derivatives on I_κ uniformly bounded for z in any compact set. Consequently, we can use Lemma 6.5 with h_2 playing the role of these functions, and we easily get, assuming $\mathcal{P}(a)$ (which in particular guarantees the condition (6.8) in Lemma 6.5) that

$$b_N(\xi) = \mathrm{O}\left(N^{-1+2a+\varepsilon}\right), \tag{6.26}$$

for any $\varepsilon > 0$, uniformly for z in any compact set of $\mathbb{C}$. Here we also used (6.2) to control the non-compact regime. From (6.12) we also have

$$\partial_{\bar{\xi}} b_N(\xi) = \mathrm{O}\left(\frac{N^{-1+2a+\varepsilon}(\Im(\xi))^2}{|(A-\xi)(B-\xi)|^{1/2}}\right)$$

in the integration regime in (6.24). By the estimates (6.25) and (6.26) we finally obtain that both error terms in (6.24) are $\mathrm{O}(N^{-1+2a+\varepsilon}\delta^{5/2})$. We proved that the right hand side of (6.22) and (6.23) together have a size bounded by $CN^{\varepsilon}\left(\frac{1}{N\delta^2}+\frac{N^{2a}\delta^{5/2}}{N}\right)$. If $\eta \in (N^{-a}, N^{-a/2})$ we choose $\delta = \eta/2$, which yields an error term at most $CN^{\varepsilon}/(N\eta^2)$. If $\eta \in (N^{-a/2}, 1)$ we choose $\delta = N^{-a/2}/2$, which yields an error at most $CN^{-1+a+\varepsilon}$. This shows that $\mathcal{P}(a/2)$ holds and it concludes the proof. □

An immediate consequence of Lemmas 6.5 and 6.6 is the following concentration of linear statistics.

Corollary 6.7. *Let V be $\mathscr{C}^4$, regular such that the equilibrium density ϱ_V is supported on a single interval $[A, B]$ and satisfy (2.3), (2.4). Let h be a $\mathscr{C}^2$ function such that $\|h\|_\infty, \|h'\|_\infty, \|h''\|_\infty < \infty$. Then for any $\varepsilon > 0$ there exists a constant $c > 0$ such that, for any $N \geqslant 1$, we have*

$$\mathbb{P}^{\mu^{(N)}}\left(\left|\sum_{k=1}^{N} h(\lambda_k) - N\int h(s)\varrho(s)\mathrm{d}s\right| > N^{\varepsilon}\right) \leqslant e^{-N^c}. \tag{6.27}$$

As we mentioned in the proof of Lemma 6.6, for fixed z, as V is $\mathscr{C}^4$, the functions $s \mapsto \Re\left(\frac{V'(E)-V'(s)}{z-s}\right)$, $s \mapsto \Im\left(\frac{V'(E)-V'(s)}{z-s}\right)$ have their first two derivatives uniformly bounded for z in any compact set. Consequently, using Corollary 6.7 and Lemma 6.2, we have, for any $\varepsilon > 0$,

$$\int_{\mathbb{R}} \frac{V'(E)-V'(s)}{z-s}\left(\varrho_1^{(N)}(s)-\varrho(s)\right)\mathrm{d}s = \mathrm{O}\left(\frac{N^{\varepsilon}}{N}\right) \tag{6.28}$$

uniformly for z in any compact set of $\mathbb{C}$. This estimate will be useful in the Subsection 6.4.

6.3 Proof of Proposition 6.2

For the proofs of Propositions 6.2 and 6.3 we will assume that $k \leqslant N/2$, thus $k = \widehat{k}$ and we remove the hat from the indices.

6.3.1 Convexification This paragraph modifies the original measure $\mu^{(N)}$ into a log-concave one, without changing the rigidity properties. This convexification first appeared in [9]. We state the main steps hereafter for the sake of completeness, and because the explicit form of the convexified measure will be required in the next multiscale analysis, subsection 6.3.2.

Let θ be a continuous nonnegative function with $\theta = 0$ on $[-1, 1]$ and $\theta'' \geqslant 1$ for $|x| > 1$. We can take for example $\theta(x) = (x-1)^2\mathbb{1}_{x>1} + (x+1)^2\mathbb{1}_{x<-1}$ in the following.

Definition 6.8. *For any fixed $s, \ell > 0$, independent of N, define the Gibbs probability measure*

$$\mathrm{d}\nu^{(s,\ell,N,c_1,\varepsilon)} = e^{-\beta N\mathcal{H}_\nu} := \frac{1}{Z^{(s,\ell)}} e^{-\beta N\psi^{(s)} - \beta N\sum_{i,j}\psi_{i,j} - \beta N(W+1)\sum_{\alpha=1}^{\ell} X_\alpha^2}\mathrm{d}\mu,$$

with Hamiltonian

$$\mathcal{H}_\nu = \psi^{(s)} + \sum_{i,j} \psi_{i,j} + (W+1)\sum_{\alpha=1}^{\ell} X_\alpha^2 + \sum_{k=1}^{N} \frac{1}{2} V(\lambda_k) - \frac{1}{N} \sum_{1\leqslant i<j\leqslant N} \log(\lambda_j - \lambda_i), \tag{6.29}$$

where

- W *is the constant appearing in the lower bound (2.3);*
- *the function* g_α *is chosen such that* $\|g_\alpha\|_\infty + \|g_\alpha'\|_\infty + \|g_\alpha''\|_\infty < \infty$ *and, for any* N *and* $k \in [\![1, N]\!]$,

$$g_\alpha'(\widetilde{\gamma}_k) = \sqrt{2}\cos\left(2\pi\left(k - \frac{1}{2}\right)\frac{\alpha}{2N}\right),$$

  where $\widetilde{\gamma}_k$ *is defined by* $\int_{-\infty}^{\widetilde{\gamma}_k} \varrho_V(s)\mathrm{d}s = \frac{1}{N}(k - \frac{1}{2})$;
- $X_\alpha = N^{-1/2}\sum_j \left(g_\alpha(\lambda_j) - g_\alpha(\widetilde{\gamma}_j)\right)$;
- $\psi^{(s)}(\lambda) = N\theta\left(\frac{s}{N}\sum_{i=1}^{N}(\lambda_i - \widetilde{\gamma}_i)^2\right)$;
- $\psi_{i,j}(\lambda) = \frac{1}{N}\theta\left(\sqrt{c_1\, N\, Q_{i,j}}(\lambda_i - \lambda_j)\right)$, *where* c_1 *is a positive constant (to be chosen large enough but independent of* N *in the next Lemma 6.9) and* Q_{ij} *is defined in the following way. Let the function* $m(n)$ *be defined on* $\mathbb{Z}$ *by* $m(n) \in [\![-N+1, N]\!]$ *and* $m(n) \equiv n \bmod(2N)$*; let* $d(k,\ell) = |m(k-l)|$ *and* $\varepsilon > 0$ *be a fixed small parameter; let*

$$R_{k,\ell} = \frac{1}{N}\frac{\varepsilon^{2/3}}{\frac{d(k,\ell)^2}{N^2} + \varepsilon^2}$$

  for any $k, \ell \in [\![-N+1, N]\!]$*;* $Q = Q(\varepsilon)$ *is then finally defined, for* $i, j \in [\![1, N]\!]$*, by*

$$Q_{i,j} = R_{i,j} + R_{1-i,j} + R_{i,1-j} + R_{1-i,1-j}.$$

Note that the measure $\nu^{(s,\ell,N,c_1,\varepsilon)}$ *depends on all five parameters but we will take the liberty to omit some or all of them in formulas where they are irrelevant.*

Thanks to these linear statistics X_α, the convexity of ν is improved compared to the one of μ, in particular the following result was proved as Lemma 3.5 in [9][2].

Lemma 6.9. *For any* $C > 0$ *there are constants* $\ell, s, c_1, \varepsilon > 0$ *depending only on* V *and* C*, such that for* N *large enough* $\nu = \nu^{(s,\ell,N,c_1,\varepsilon)}$ *satisfies, for any* $\mathbf{v} \in \mathbb{R}^N$,

$$\langle \mathbf{v}, (\nabla^2\mathcal{H}_\nu)\mathbf{v}\rangle \geqslant C\,\|\mathbf{v}\|^2.$$

An important fact for the measure $\nu^{(s,\ell)}$ is that it does not deviate much from μ concerning events with very small probability. More precisely, the following result holds.

We say that a sequence of events $(A_N)_{N\geqslant 1}$ is *exponentially small* for a sequence of probability measures $(m_N)_{N\geqslant 1}$ if there are constants $C, c > 0$ such that, for any N, we have

$$m_N(A_N) \leqslant Ce^{-N^c}.$$

[2]Note that in Lemma 3.5 in [9], the constant c was just required to be positive but following the reasoning in [9] it can be made arbitrary large first by choosing M in [9, Equation (3.1)] sufficiently large.

Lemma 6.10. *For any fixed choice of the parameters $s, \ell, c_1, \varepsilon$ defining $\nu^{(N)}$, the measures $(\mu^{(N)})_{N\geqslant 1}$ and $(\nu^{(N)})_{N\geqslant 1}$ have the same exponentially small events. In particular, for any $a>0$, concentration at scale a for $(\mu^{(N)})_{N\geqslant 1}$ is equivalent to concentration at scale a for $(\nu^{(N)})_{N\geqslant 1}$.*

Proof. The first statement can be proved as Lemma 3.6 in [9], except that in that paper we used $\mathbb{E}(NX_\alpha^2) < (\log N)^2$, an estimate true in the context of an analytic potential V. Here we only assume that V is $\mathscr{C}^4$; then by Lemma 6.5, for any $\varepsilon > 0$ and for large enough N, we have $\mathbb{E}(NX_\alpha^2) \leqslant N^{\varepsilon}$. As ε is arbitrarily small the remainder of the proof goes in the same way as Lemma 3.6 in [9].

Note that the second statement of the lemma is not a completely direct application of the first one: if rigidity at scale a holds for $(\mu^{(N)})_{N\geqslant 1}$, by using the first statement we obtain that for any $\varepsilon > 0$ there are $c > 0$ and N_0 such that for all $N \geqslant N_0$ and $k \in [\![1, N]\!]$

$$\mathbb{P}^{\nu^{(N)}}\left(|\lambda_k - \mathbb{E}^{\mu^{(N)}}\lambda_k| > N^{-\frac{2}{3}+a+\varepsilon}(\widehat{k})^{-\frac{1}{3}}\right) \leqslant e^{-N^c}. \tag{6.30}$$

However, to obtain concentration for $\nu^{(N)}$, we need to estimate the difference $\mathbb{E}^{\mu^{(N)}}\lambda_k - \mathbb{E}^{\nu^{(N)}}\lambda_k$.

We know from (6.7) that for any $\kappa > 0$, there is a $C > 0$ such that for $x \notin [A-\kappa, B+\kappa]$, we have

$$\varrho_1^{(N,\mu)}(x) \leqslant (|x|+1)^{-CN}. \tag{6.31}$$

Similarly to Lemma 3.6 in [9], for any $\varepsilon > 0$ there is a $c > 0$ such that for any event A,

$$\mathbb{P}_\nu(A) \leqslant e^{cN^\varepsilon}\, \mathbb{P}_\mu(A). \tag{6.32}$$

Equations (6.31) and (6.32) imply that for some positive constants c and c',

$$\varrho_1^{(N,\nu)}(x) \leqslant (|x|+1)^{-cN} e^{c'N^\varepsilon}. \tag{6.33}$$

Equation (6.30) together with the large-deviation type estimate (6.33) imply that

$$|\mathbb{E}^{\mu^{(N)}}(\lambda_k) - \mathbb{E}^{\nu^{(N)}}(\lambda_k)| = \mathrm{O}(N^{-\frac{2}{3}+a+\varepsilon}(\widehat{k})^{-\frac{1}{3}}),$$

and subsequently that concentration holds for $\nu^{(N)}$ at scale a. That concentration for ν implies concentration for μ can be proved in a similar way (it is easier because (6.32) is not needed, the necessary decay follows directly from (6.31)). □

6.3.2 The multiscale analysis This subsection is similar to subsection 3.2 in [8], but we adapted the arguments in the scalings to improve the rigidity scale up to the edges.

In this subsection, $s, \ell, c_1, \varepsilon$ are chosen so that $\nu^{(N)}$ satisfies the convexity relation from Lemma 6.9 with $C = 10W$. We now define the locally constrained measures, up to the edge; these measures ensure strict convexity bounds when knowing rigidity at scale a.

Definition 6.11. *Let $\varepsilon > 0$. For any given $k \in [\![1, N]\!]$ and any integer $1 \leqslant M \leqslant N/2$, we denote*

$$I^{(k,M)} = \begin{cases} [\![k, k+M-1]\!] & \textit{if} \quad k \leqslant N/2 \\ [\![k-M+1, k]\!] & \textit{if} \quad k \geqslant N/2 \end{cases}.$$

Moreover, let

$$\phi^{(k,M)} = \sum_{i<j, i,j\in I^{(k,M)}} \theta\left(\frac{N^{\frac{2}{3}-\varepsilon}(\hat{k})^{\frac{1}{3}}}{M}(\lambda_i - \lambda_j)\right).$$

We define the probability measure

$$\mathrm{d}\omega^{(k,M)} := \frac{1}{Z} e^{-\beta\phi^{(k,M)}} \mathrm{d}\nu, \tag{6.34}$$

where $Z = Z_{\omega^{(k,M)}}$. *The measure* $\omega^{(k,M)}$ *will be referred to as locally constrained transform of* ν, *around* k, *with width* M. *The dependence of the measure on* ε *will be suppressed in the notation.*

We will also frequently use the following notation for block averages in any sequence $(x_i)_i$:

$$x_k^{[M]} := \frac{1}{M} \sum_{i \in I^{(k,M)}} x_i.$$

The reason for introducing these locally constrained measures is that they improve the convexity in $I^{(k,M)}$ on the subspace orthogonal to the constants, as explained in the following lemma which is a slight modification of Lemma 3.8 of [8].

Lemma 6.12. *Write the probability measure* $\omega^{(k,M)}$ *from* (6.34) *as* $\omega^{(k,M)} = \frac{1}{Z} e^{-\beta N(\mathcal{H}_1+\mathcal{H}_2)}\mathrm{d}\lambda$, *where we denote*

$$\mathcal{H}_1 := \frac{1}{N}\phi^{(k,M)} - \frac{1}{2N} \sum_{i<j, i,j \in I^{(k,M)}} \log|\lambda_i - \lambda_j|,$$

$$\mathcal{H}_2 := \mathcal{H}_\nu + \frac{1}{2N} \sum_{i<j, i,j \in I^{(k,M)}} \log|\lambda_i - \lambda_j|.$$

Then $\nabla^2\mathcal{H}_2 \geqslant 0$ *and denoting* $\mathbf{v} = (v_i)_{i \in I^{(k,M)}}$, *we also have*

$$\langle \mathbf{v}, (\nabla^2\mathcal{H}_1)\mathbf{v}\rangle \geqslant \frac{1}{2N}\left(\frac{N^{\frac{2}{3}-\varepsilon}(\hat{k})^{\frac{1}{3}}}{M}\right)^2 \sum_{i,j \in I^{(k,M)}, i<j} (v_i - v_j)^2.$$

Proof. Note that in the modification $\mathcal{H}_2$ of $\mathcal{H}_\nu$, we only removed half of the pairwise interactions[3] between the λ's in $I^{(k,M)}$. This allows us to use Lemma 6.9 (with the choice $c = 10W$) to prove the convexity of $\mathcal{H}_2$. Denoting $\mathcal{V} = \mathcal{V}(\boldsymbol{\lambda}) := \frac{1}{2}\sum_j V(\lambda_j)$, we indeed have

$$\nabla^2\mathcal{H}_2 = \nabla^2(\mathcal{H}_2 - \frac{1}{2}\mathcal{V}) + \frac{1}{2}\nabla^2\mathcal{V} \geqslant \frac{1}{2}\nabla^2\mathcal{H}_\nu + \frac{1}{2}\nabla^2\mathcal{V} \geqslant \frac{1}{2}10W - \frac{1}{2}W \geqslant 0.$$

In the first inequality we used that

$$\mathcal{H}_2 - \frac{1}{2}\mathcal{V} - \frac{1}{2}\mathcal{H}_\nu = \frac{1}{2}\Big(\psi^{(s)} + \sum_{i,j}\psi_{i,j} + (W+1)\sum_{\alpha=1}^{\ell} X_\alpha^2\Big)$$

from (6.29) and each term on the right hand side is convex by their explicit definitions.

Concerning the lower bound for $\nabla^2\mathcal{H}_1$, a simple calculation gives

$$\langle \mathbf{v}, (\nabla^2\mathcal{H}_1)\mathbf{v}\rangle \geqslant \frac{1}{2N}\sum_{i<j, i,j\in I^{(k,M)}} (v_i - v_j)^2 \left(\frac{1}{(\lambda_i-\lambda_j)^2} + \left(\frac{N^{\frac{2}{3}-\varepsilon}(\hat{k})^{\frac{1}{3}}}{M}\right)^2 \mathbb{1}\Big\{|\lambda_i - \lambda_j| > \frac{M}{N^{\frac{2}{3}-\varepsilon}(\hat{k})^{\frac{1}{3}}}\Big\}\right),$$

which concludes the proof. □

[3]This minor point was not made explicit in [9].

The above convexity bound on $\mathcal{H}_1$ allows us to get an improved concentration for functions depending on differences between particles, as shown in the following lemma:

Lemma 6.13 (Lemma 3.9 in [8])**.** *Decompose the coordinates $\lambda = (\lambda_1, \dots, \lambda_N)$ of a point in $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^{N-m}$ as $\boldsymbol{\lambda} = (x, y)$, where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^{N-m}$. Let $\omega = \frac{1}{Z} e^{-N\mathcal{H}}$ be a probability measure on $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^{N-m}$ such that $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$, with $\mathcal{H}_1 = \mathcal{H}_1(x)$ depending only on the x variables and $\mathcal{H}_2 = \mathcal{H}_2(x, y)$ depending on all coordinates. Assume that, for any $\lambda \in \mathbb{R}^N$, $\nabla^2 \mathcal{H}_2(\lambda) \geqslant 0$. Assume moreover that $\mathcal{H}_1(x)$ is independent of $x_1 + \dots + x_m$, i.e., $\sum_{i=1}^m \partial_i \mathcal{H}_1(x) = 0$ and that for any $x, v \in \mathbb{R}^m$,*

$$\langle \mathbf{v}, (\nabla^2 \mathcal{H}_1(x)) \mathbf{v} \rangle \geqslant \frac{\xi}{m} \sum_{i,j=1}^m |v_i - v_j|^2$$

with some positive $\xi > 0$. Then for any function of the form $f(\lambda) = F(\sum_{i=1}^m v_i x_i)$, where $\sum_i v_i = 0$ and $F : \mathbb{R} \to \mathbb{R}$ is any smooth function, we have

$$\int f^2 \log f^2 \mathrm{d}\omega - \left(\int f^2 \mathrm{d}\omega \right) \log \left(\int f^2 \mathrm{d}\omega \right) \leqslant \frac{1}{\xi N} \int |\nabla f|^2 \mathrm{d}\omega.$$

A direct application of Lemmas 6.12 and 6.13 gives, by Herbst's lemma, the following concentration estimate.

Corollary 6.14. *For any function $f(\{\lambda_i, i \in I^{(k,M)}\}) = \sum_{I^{(k,M)}} v_i \lambda_i$ with $\sum_i v_i = 0$ and for any $u > 0$ we have, for some constant $c > 0$ that depends only on β and V,*

$$\mathbb{P}^{\omega^{(k,M)}}(|f - \mathbb{E}^{\omega^{(k,M)}}(f)| > u) \leqslant 2 \exp\left(-\frac{c}{M|v|^2} \left(N^{\frac{2}{3}-\varepsilon} (\hat{k})^{\frac{1}{3}} \right)^2 u^2 \right).$$

When the function f is chosen of type $\lambda_k^{[M_1]} - \lambda_k^{[M]}$, we get in particular the following concentration.

Lemma 6.15. *Take any $\varepsilon > 0$. There are constants $c > 0$, N_0 such that for any $N \geqslant N_0$, any integers $1 \leqslant M_1 \leqslant M \leqslant N/2$, any $k \in [\![1, N]\!]$, and $\omega^{(k,M)}$ from Definition 6.11 associated with k, M, ε, we have for any $u > 0$,*

$$\mathbb{P}^{\omega^{(k,M)}} \left(\left| \lambda_k^{[M_1]} - \lambda_k^{[M]} - \mathbb{E}^{\omega^{(k,M)}} \left(\lambda_k^{[M_1]} - \lambda_k^{[M]} \right) \right| > u (N^{\frac{2}{3}-\varepsilon} (\hat{k})^{\frac{1}{3}})^{-1} \sqrt{\frac{M}{M_1}} \right) \leqslant e^{-cu^2}.$$

Proof. Relying on Corollary 6.14, writing $\lambda_k^{[M_1]} - \lambda_k^{[M]} = \sum v_i \lambda_i$ with some constants v_i, one only needs to prove $|v|^2 \leqslant 1/M_1$ to conclude. An explicit computation gives $|v|^2 = 1/M_1 - 1/M$. □

The following three Lemmas are slight modifications of Lemmas 3.15, 3.16 and 3.17 from [8].

Lemma 6.16. *Assume that for μ rigidity at scale a holds. Take arbitrary $\varepsilon > 0$. There exist constants $c, N_0 > 0$ such that for any $N \geqslant N_0$, any integer M satisfying $N^a \leqslant M \leqslant N/2$, any $k, j \in [\![1, N]\!]$ we have*

$$|\mathbb{E}^{\nu}(\lambda_j) - \mathbb{E}^{\omega^{(k,M)}}(\lambda_j)| \leqslant e^{-N^c},$$

where the measure $\omega^{(k,M)}$ is given by Definition 6.11 with parameters k, M, ε.

Proof. Note that $\theta(x)=0$ if $|x|<1$, so if the $\phi^{(k,M)}$ term in the definition of ω is non-zero then either $|\lambda_k-\gamma_k|$ or $|\lambda_{k+M}-\gamma_{k+M}|$ is greater than $\frac{1}{3}MN^{-\frac{2}{3}+\varepsilon}k^{-1/3}$, where we used that $\gamma_{k+M}-\gamma_k\leqslant\frac{1}{3}MN^{-\frac{2}{3}+\varepsilon}k^{-1/3}$. Since rigidity at scale a holds for μ, it also holds for ν by Lemma 6.10, so both events have exponentially small probability (remember that $M\geqslant N^a$). This easily implies that $\int e^{-\beta\phi^{(k,M)}}\mathrm{d}\nu>1/2$ for large enough N, and therefore $\mathbb{P}^{\omega^{(k,M)}}(A)\leqslant 2\mathbb{P}^{\nu}(A)$ for any event A. Consequently (6.33) holds when replacing ν by ω:

$$\varrho_1^{(N,\omega^{(k,M)})}(x)\leqslant 2(|x|+1)^{-cN}e^{c'(\log N)^2} \tag{6.35}$$

for some constants c,c'. The total variation norm is bounded by the square root of the entropy (defined for a probability measure ν and a probability density f (w.r.t. ν), by $S_\nu(f)=\int f\log f\mathrm{d}\nu$); moreover, by (6.33) and (6.35) the particles are bounded with very high probability, both for the measure ν and $\omega^{(k,M)}$. We therefore have

$$|\mathbb{E}^{\nu}(\lambda_j)-\mathbb{E}^{\omega^{(k,M)}}(\lambda_j)|\leqslant C\sqrt{S_{\omega^{(k,M)}}(\mathrm{d}\nu/\mathrm{d}\omega^{(k,M)})}+\mathrm{O}(e^{-cN})$$

for some $c,C>0$ independent of N,k,j. In order to bound this entropy, note that the measure ν satisfies a logarithmic Sobolev inequality with constant of order N (this follows from the convexity estimate obtained in Lemma 6.9 and an application of the Bakry-Émery criterion [4]): for any smooth $f\geqslant 0$ with $\int f\mathrm{d}\nu=0$, we have

$$\int f\log f\mathrm{d}\nu\leqslant\frac{1}{cN}\int|\nabla\sqrt{f}|^2\mathrm{d}\nu, \tag{6.36}$$

for some small fixed $c>0$. We therefore obtain, for some large fixed $C>0$,

$$S_{\omega^{(k,M)}}(\mathrm{d}\nu/\mathrm{d}\omega^{(k,M)})\leqslant N^C\,\mathbb{E}^{\nu}\left(\theta'\left(\frac{(\lambda_{k+M}-\lambda_k)N^{\frac{2}{3}-\varepsilon}(\hat{k})^{\frac{1}{3}}}{M}\right)^2\right). \tag{6.37}$$

We claim that the above expectation can be bounded by e^{-N^c} for some fixed $c>0$ if N is large. To prove this exponential bound, we assume $k<N/2$ for simplicity. As we saw at the beginning of this proof, if the above θ' term is non-zero then either $|\lambda_k-\gamma_k|$ or $|\lambda_{k+M}-\gamma_{k+M}|$ is greater than $\frac{1}{3}MN^{-\frac{2}{3}+\varepsilon}k^{-1/3}$, and both events have exponentially small probability. Together with $\theta'(x)^2<4x^2$ and (6.33), this proves the desired estimate (6.37). ☐

Lemma 6.17. *Assume that for μ rigidity at scale a holds. Take arbitrary $\varepsilon>0$. There are constants $c>0$ and N_0 such that for any $N\geqslant N_0$, any integers $N^a\leqslant M\leqslant N/2$, $1\leqslant M_1\leqslant M$, and $k\in[\![1,N]\!]$, we have*

$$\mathbb{P}^{\nu}\left(\left|\lambda_k^{[M_1]}-\lambda_k^{[M]}-\mathbb{E}^{\nu}\left(\lambda_k^{[M_1]}-\lambda_k^{[M]}\right)\right|>(N^{\frac{2}{3}-\varepsilon}(\hat{k})^{\frac{1}{3}})^{-1}\sqrt{\frac{M}{M_1}}\right)\leqslant e^{-N^c}.$$

Proof. By Lemma 6.15 we know that the result holds when considering $\omega^{(k,M)}$ instead of ν. Moreover, by Lemma 6.16 the difference

$$\left|\mathbb{E}^{\nu}\left(\lambda_k^{[M_1]}-\lambda_k^{[M]}\right)-\mathbb{E}^{\omega^{(k,M)}}\left(\lambda_k^{[M_1]}-\lambda_k^{[M]}\right)\right|$$

is exponentially small. So we just need to prove that

$$(\mathbb{P}^{\nu}-\mathbb{P}^{\omega^{(k,M)}})\left(\left|\lambda_k^{[M_1]}-\lambda_k^{[M]}-\mathbb{E}^{\nu}\left(\lambda_k^{[M_1]}-\lambda_k^{[M]}\right)\right|>(N^{\frac{2}{3}-\varepsilon}(\hat{k})^{\frac{1}{3}})^{-1}\sqrt{\frac{M}{M_1}}\right)$$

is bounded by e^{-N^c}. This is true because $|\mathbb{P}^{\nu}(A)-\mathbb{P}^{\omega^{(k,M)}}(A)|$ is bounded by $(S_{\omega^{(k,M)}}(\mathrm{d}\nu/\mathrm{d}\omega^{(k,M)}))^{1/2}$, which is exponentially small, as proved below (6.37). ☐

Lemma 6.18. *Assume that for μ rigidity at scale a holds. For any $\varepsilon>0$, there are constants $c, N_0>0$ such that for any $N\geqslant N_0$ and $k\in[\![1,N]\!]$, we have*

$$\mathbb{P}^\nu\left(\left|\lambda_k-\lambda_k^{[N/2]}-\mathbb{E}^\nu(\lambda_k-\lambda_k^{[N/2]})\right|>N^{-\frac{2}{3}+\frac{a}{2}+\varepsilon}(\hat{k})^{-\frac{1}{3}}\right)\leqslant e^{-N^c}.$$

Proof. Note first that

$$\left|\lambda_k-\lambda_k^{[N/2]}-\mathbb{E}^\nu(\lambda_k-\lambda_k^{[N/2]})\right|\leqslant\left|\lambda_k-\lambda_k^{[N^a]}-\mathbb{E}^\nu(\lambda_k-\lambda_k^{[N^a]})\right|+\left|\lambda_k^{[N^a]}-\lambda_k^{[N/2]}-\mathbb{E}^\nu(\lambda_k^{[N^a]}-\lambda_k^{[N/2]})\right|.$$

By the choice $M_1=1$, $M=N^a$ in Lemma 6.17, the ν-probability that the first term is greater than $N^{-\frac{2}{3}+\frac{a}{2}+\varepsilon}(\hat{k})^{-\frac{1}{3}}$ is exponentially small, uniformly in k, as desired. Concerning the second term, given some $r>0$ and $q\in\mathbb{N}$ defined by $1-r\leqslant a+qr<1$, it is bounded by

$$\sum_{\ell=0}^{q-1}\left|\lambda_k^{[N^{a+(\ell+1)r}]}-\lambda_k^{[N^{a+\ell r}]}-\mathbb{E}^\nu\left(\lambda_k^{[N^{a+(\ell+1)r}]}-\lambda_k^{[N^{a+\ell r}]}\right)\right|+\left|\lambda_k^{[a+qr]}-\lambda_k^{[N/2]}-\mathbb{E}^\nu\left(\lambda_k^{[N^{a+qr}]}-\lambda_k^{[N/2]}\right)\right|.$$

By Lemma 6.17, for any $\varepsilon>0$, each one of these $q+1$ terms has an exponentially small probability of being greater than $N^{-\frac{2}{3}+\varepsilon+\frac{r}{2}}(\hat{k})^{-\frac{1}{3}}$. Consequently, choosing any ε and r (and therefore q) such that $\varepsilon+\frac{r}{2}<a/2$ concludes the proof. □

Proof of Proposition 6.2. Obviously,

$$|\lambda_k-\mathbb{E}^\nu(\lambda_k)|\leqslant|\lambda_k-\lambda_k^{[N/2]}-\mathbb{E}^\nu(\lambda_k-\lambda_k^{[N/2]})|+|\lambda_k^{[N/2]}-\mathbb{E}^\nu(\lambda_k^{[N/2]})|.$$

By Lemma 6.18, the first term has exponentially small probability to be greater than $N^{-\frac{2}{3}+\frac{a}{2}+\varepsilon}(\hat{k})^{-\frac{1}{3}}$. Moreover, as ν satisfies (6.36), by the classical Herbst's lemma (see e.g. [2]), the second term has exponentially small probability to be greater than $N^{-1+\varepsilon}$. This concludes the proof of concentration at scale $a/2$ for the measure ν.

Consequently, by Lemma 6.10, for any $\varepsilon>0$, there are constants $c, N_0>0$ such that for any $N\geqslant N_0$ and $k\in[\![1,N]\!]$ we have

$$\mathbb{P}^\mu\left(|\lambda_k-\mathbb{E}^\nu\lambda_k|>N^{-\frac{2}{3}+\frac{a}{2}+\varepsilon}(\hat{k})^{-\frac{1}{3}}\right)\leqslant\ e^{-N^c}.$$

This probability bound together with (6.31) implies that

$$|\mathbb{E}^\nu\lambda_k-\mathbb{E}^\mu\lambda_k|=\mathrm{O}(N^{-\frac{2}{3}+\frac{a}{2}+\varepsilon}(\hat{k})^{-\frac{1}{3}})$$

uniformly in N and k, and concludes the proof of concentration at scale $a/2$ for μ. □

6.4 Proof of Proposition 6.3

We aim at improving the accuracy from scale a to scale $11a/12$, now that we know concentration at scale $a/2$ from the proven Proposition 6.2. In [8] and [9] we proved that, in the bulk of the spectrum,

$$m_N(z)-m(z)\sim\mathrm{Var}_N(z),\qquad \mathrm{Var}_N(z):=\frac{1}{N^2}\mathrm{Var}_{\mu^{(N)}}\left(\sum_{k=1}^N\frac{1}{z-\lambda_k}\right).$$

Concentration at scale $a/2$ then allowed us to properly bound the above variance term, which yielded good estimates on m_N-m and therefore an improved accuracy. In these previous works analyticity of V was essential, as it was in [36] and [50].

We first explain the method for the accuracy improvement, for non-analytic V. The following modification of the loop equation will be useful: from the difference of (6.16) (with $h=0$) and (6.17) we obtain (noting $z=E+\mathrm{i}\eta$)

$$(m_N(z)-m(z))^2+(2m(z)-V'(E))(m_N(z)-m(z))+\int_{\mathbb{R}}\frac{V'(E)-V'(s)}{z-s}(\varrho_1^{(N)}(s)-\varrho(s))\mathrm{d}s \\ -\frac{1}{N}\left(\frac{2}{\beta}-1\right)m_N'(z)-\mathrm{Var}_N(z)=0. \quad (6.38)$$

In the above equation, the integral term can be neglected thanks to ((6.28)). The $(m_N-m)^2$ and $N^{-1}m_N'$ terms are easily shown to be of negligible order too, so for z close to $[A,B]$ we have

$$(2m(z)-V'(E))(m_N(z)-m(z))\sim \mathrm{Var}_N(z).$$

For z close to the bulk of the spectrum, $2m(z)-V'(E)$ is bounded away from 0, so this equation yields an accurate upper bound on m_N-m.

The rest of the proof of accuracy improvement involves a major technical difficulty: optimal estimates up to the edge are difficult to obtain, because $2m(z)-V'(E)$ vanishes when z is close to A or B. As a main difference from the accuracy improvement in [8], our current use of the loop equation will allow finer estimates, improving accuracy of *one* given particle (the first one), in Lemma 6.25. The accuracy improvement for both extreme particles together with the amelioration for λ_k's with $\hat{k}\geqslant N^{3a/4}$ will imply improvement for all particles. The following series of lemmas makes these heuristics rigorous.

For any $A<E<B$ we define

$$\kappa_E=\min(|E-A|,|E-B|)$$

the distance of E from the edges of the support of the equilibrium measure. Also, in this section, $a(N)\ll b(N)$ means $a(N)=\mathrm{o}(b(N))$ as $N\to\infty$. We will finally use the notations

$$\Sigma_{\mathrm{Int}}^{(N)}(u,\tau):=\left\{z=E+\mathrm{i}\eta: A\leqslant E\leqslant B,\ N^{-1+u}\kappa_E^{-1/2}\leqslant\eta\leqslant\tau\right\},$$
$$\Sigma_{\mathrm{Ext}}^{(N)}(u,\tau):=\left\{z=E+\mathrm{i}\eta: E\in[A-2N^{-2/3+u},A-N^{-2/3+u}],\ N^{-2/3+u}\leqslant\eta\leqslant\tau\right\}.$$

Lemma 6.19. *Assume that*

$$\mathrm{Var}_N(z)\ll\max(\kappa_E,\eta) \quad (6.39)$$

as $N\to\infty$, uniformly in $\Sigma_{\mathrm{Int}}^{(N)}(u,\tau)$, for some fixed $u>0$ and small $\tau>0$. Then for any $\varepsilon>0$ there are constants C, $0<\delta<\tau$ such that for any $z\in\Sigma_{\mathrm{Int}}^{(N)}(u,\delta)$ we have

$$|m_N(z)-m(z)|\leqslant C\left(\frac{N^{\varepsilon}}{N\eta}+\frac{|\mathrm{Var}_N(z)|}{\max(\sqrt{\kappa_E},\sqrt{\eta})}\right).$$

The same statement holds when replacing $\Sigma_{\mathrm{Int}}^{(N)}$ everywhere by $\Sigma_{\mathrm{Ext}}^{(N)}$.

Proof. We first note that

$$\frac{1}{N}|m_N'(z)|=\frac{1}{N^2}\left|\mathbb{E}^{\mu}\sum_j\frac{1}{(z-\lambda_j)^2}\right| \\ \leqslant\frac{1}{N\eta}\Im m_N(z)\leqslant\frac{1}{N\eta}|m_N(z)-m(z)|+\frac{1}{N\eta}|\Im m(z)|\leqslant\frac{1}{N\eta}|m_N(z)-m(z)|+\frac{C}{N\eta}\max(\sqrt{\kappa_E},\sqrt{\eta}),$$

where we used $\Im m(z) \leqslant C \max\{\sqrt{\kappa_E}, \sqrt{\eta}\}$, an easy estimate due to the square root singularity of the equilibrium measure ϱ on the edges. Equation (6.38) therefore implies

$$\begin{aligned}
&(m_N(z) - m(z))^2 + b(z)(m_M(z) - m(z)) + c(z) = 0, \qquad (6.40)\\
&b(z) := 2m(z) - V'(E) + \frac{c_1(z,N)}{N\eta},\\
&c(z) := \frac{c_2(z,N)}{N\eta} \max(\sqrt{\kappa_E}, \sqrt{\eta}) + c_3(z,N) N^{\varepsilon-1} - \mathrm{Var}_N(z),
\end{aligned}$$

where there is a constant $C > 0$ such that for any for any N and z, $|c_1(z,N)|, |c_2(z,N)|, |c_3(N,z)| < C$ (we used (6.28) to bound the integral term in (6.38)).

To solve the above quadratic equation (6.40), we need a priori estimates on the coefficients. As ϱ has a square root singularity close to the edges, there is a constant $c > 0$ such that

$$c \max(\sqrt{\kappa_E}, \sqrt{\eta}) < |2m(z) - V'(E)| < c^{-1} \max(\sqrt{\kappa_E}, \sqrt{\eta}). \qquad (6.41)$$

On the other hand, unifomly in $\Sigma^{(N)}_{\mathrm{Int}}(u,\tau)$ we have

$$\frac{1}{N\eta} \ll \max(\sqrt{\kappa_E}, \sqrt{\eta}), \qquad (6.42)$$

so we obtain

$$|b(z)| \gg \max(\sqrt{\kappa_E}, \sqrt{\eta}). \qquad (6.43)$$

Moreover, from (6.39) and (6.42), the estimate

$$c(z) \ll \max(\kappa_E, \eta) \qquad (6.44)$$

holds. From the estimates (6.43) and (6.44) we have $b(z)^2 \gg c(z)$, so the quadratic equation (6.40) yields

$$m_N(z) - m(z) = \frac{-b(z) \pm \sqrt{b(z)^2 - 4c(z)}}{2} \underset{N\to\infty}{\sim} \frac{1}{2}\left(-b(z) \pm b(z)\left(1 - \frac{4c(z)}{2b(z)^2}\right)\right).$$

For E in the bulk and $\eta \sim 1$ we know that $m_N(z) - m(z) \to 0$ and $b(z) \sim 1$, so the appropriate asymptotics needs to be $m_N(z) - m(z) \sim -c(z)/b(z)$. By continuity, this holds in $\Sigma^{(N)}_{\mathrm{Int}}(u,\tau)$, concluding the proof. In the case of the domain $\Sigma^{(N)}_{\mathrm{Ext}}(u,\delta)$, the proof is the same. □

The following lemma is similar to the previous one, but aims at controlling the extreme eigenvalues. For this, we introduce the notation

$$\Omega^{(N)}(d,s,\tau) = \left\{ z = E + \mathrm{i}\eta \mid \eta = N^{-\frac{2}{3}+s}, A - \tau \leqslant E \leqslant A - N^{-\frac{2}{3}+d} \right\}. \qquad (6.45)$$

Lemma 6.20. *Assume that for some $0 < d, s \leqslant 2/3$, $\tau > 0$,*

$$\mathrm{Var}_N(z) + \frac{1}{N}|m_N'(z)| \ll |z - A| \qquad (6.46)$$

uniformly on $\Omega^{(N)}(d,s,\tau)$. Then for any $\varepsilon > 0$ we have, uniformly on $\Omega^{(N)}(d,s,\tau)$, we have

$$|m_N(z) - m(z)| = \mathrm{O}\left(|z - A|^{-1/2}\left(\mathrm{Var}_N(z) + \frac{1}{N}|m_N'(z)| + N^{-1+\varepsilon}\right)\right).$$

Proof. This lemma can be proved in a way perfectly analogous to Lemma 6.19: we solve the quadratic equation (6.38), after bounding its integral term by $N^{-1+\varepsilon}$. Two solutions are possible, which have asymptotics (using (6.41) and (6.46))

$$m_N(z)-m(z)\sim\frac{\mathrm{Var}_N(z)+\frac{c_1(z,N)}{N}|m_N'(z)|+c_2(z,N)N^{-1+\varepsilon}}{2m(z)-V'(E)}\quad\text{or}\quad m_N(z)-m(z)\sim -2m(z)+V'(E),$$

where $|c_1(z,N)|,|c_2(z,N)|\leqslant C$ for some $C>0$ independent of z and N. For $z=A-\tau+\mathrm{i}N^{-\frac{2}{3}+s}$, we know that $m_N-m\to 0$ (this relies on the macroscopic convergence of the spectral measure and the large deviation estimate (6.2)). This together with the continuity of m_N-m and (6.46), (6.41), implies that the proper choice is the first one uniformly in $\Omega^{(N)}(d,s,\tau)$. □

The proofs of the following three technical lemmas are postponed to Appendix A.

Lemma 6.21. *Assume that rigidity at scale a and concentration at scale $a/2$ hold. Then for any fixed $\tau>0,\varepsilon>0$, uniformly on $\Sigma^{(N)}_{\mathrm{Int}}(3a/4+\varepsilon,\tau)$ one has*

$$\frac{1}{N^2}\mathrm{Var}\left(\sum\frac{1}{z-\lambda_i}\right)\ll\frac{N^{\frac{3a}{4}}}{N\eta}\max(\kappa_E^{1/2},\eta^{1/2}).$$

Lemma 6.22. *Assume that rigidity at scale a holds, and moreover that the extra rigidity at scale $3a/4$ holds except for a few edge particles, in the following sense: for any $\varepsilon>0$, there are constants $c,N_0>0$ such that for any $N\geqslant N_0$ and $\hat k\geqslant N^{\frac{3a}{4}+\varepsilon}$ we have*

$$\mathbb{P}^{\mu}\left(|\lambda_k-\gamma_k|>N^{-\frac{2}{3}+\frac{3a}{4}+\varepsilon}(\hat k)^{-\frac{1}{3}}\right)\leqslant\ e^{-N^c}.$$

Let $d>2a/3$ and $\tau>0$ be small enough. Then uniformly in $\Sigma^{(N)}_{\mathrm{Ext}}(d,\tau)$ one has

$$\frac{1}{N^2}\mathrm{Var}\left(\sum\frac{1}{z-\lambda_i}\right)\ll\frac{1}{N\eta}\max(\kappa_E^{1/2},\eta^{1/2}).$$

Lemma 6.23. *Assume that rigidity at scale a holds, and moreover that the extra rigidity at scale $3a/4$ holds except for a few edge particles, in the following sense: for any $\varepsilon>0$, there are constants $c,N_0>0$ such that for any $N\geqslant N_0$ and $\hat k\geqslant N^{\frac{3a}{4}+\varepsilon}$ we have*

$$\mathbb{P}^{\mu}\left(|\lambda_k-\gamma_k|>N^{-\frac{2}{3}+\frac{3a}{4}+\varepsilon}(\hat k)^{-\frac{1}{3}}\right)\leqslant e^{-N^c}.$$

Let $a>d>s>a/2$. Then uniformly in $\Omega^{(N)}(d,s,\tau)$ (defined in (6.45)) we have

$$\frac{1}{N}m_N'(z)=\mathrm{O}\left(N^{-\frac{2}{3}+\frac{3a}{4}+\varepsilon-2s}\mathbb{1}_{|z-A|<N^{-\frac{2}{3}+a+\varepsilon}}+N^{-1+\varepsilon}|z-A|^{-1/2}\right),\tag{6.47}$$

$$\frac{1}{N^2}\,\mathrm{Var}\left(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{z-\lambda_i}\right)=\mathrm{O}\left(N^{-\frac{2}{3}-4s+2a+\varepsilon}\mathbb{1}_{|z-A|<N^{-\frac{2}{3}+a+\varepsilon}}+N^{-2+a+\varepsilon}|z-A|^{-2}\right).\tag{6.48}$$

We will need to transfer information on the Stieltjes transform to the typical location of the points. The following result is similar to Lemma 2.3 in [9] for example, except that this version will be suited to take into account the weaker information on m_N-m near the edges.

Lemma 6.24. *a) Let $\widetilde{\varrho}(s)\mathrm{d}s$ be an arbitrary signed measure (depending on N) and let*

$$S(z) := \int \frac{\widetilde{\varrho}(s)}{z-s}\mathrm{d}s$$

be its Stieltjes transform. Let $\tau > 0$ be fixed, $\eta > N^{-1}$, $E \in [A, B]$ and $\eta_E = \kappa_E^{-\frac{1}{2}}\eta$. Assume that for some (possibly N-dependent) U we have

$$|S(x+\mathrm{i}y)| \leqslant \frac{U}{Ny} \text{ for any } x \in [E, E+\eta_E] \text{ and } \eta_E < y < \tau, \tag{6.49}$$

$$|S(x+\mathrm{i}y)| \leqslant \frac{U}{N} \text{ for any } x \in \mathbb{R} \text{ and } \tau/2 < y < \tau, \tag{6.50}$$

$$\text{there is a constant } L > 0 \text{ such that for any } N \text{ and } |s| > L,\ |\widetilde{\varrho}(s)| \leqslant |s|^{-cN}. \tag{6.51}$$

Define a function $f = f_{E,\eta_E}$: $\mathbb{R} \to \mathbb{R}$ such that $f(x) = 1$ for $x \in (-\infty, E]$, $f(x)$ vanishes for $x \in [E+\eta_E, \infty)$, moreover $|f'(x)| \leqslant c\,{\eta_E}^{-1}$ and $|f''(x)| \leqslant c\,{\eta_E}^{-2}$, for some constant c independent of N. Then for some constant $C > 0$, independent of N, we have

$$\left|\int f(\lambda)\widetilde{\varrho}(\lambda)\mathrm{d}\lambda\right| \leqslant C\ \frac{U(\log N)}{N}.$$

b) The same result holds for a specific value of E below A, namely for E that is the unique solution of the equation $E = A - 2\eta_E$.

Proof. We prove a), item b) is analogous. From (B.13) in [22]:

$$\left|\int_{-\infty}^{\infty} f(\lambda)\widetilde{\varrho}(\lambda)\mathrm{d}\lambda\right| \leqslant C\left|\iint y f''(x)\chi(y)\Im S(x+\mathrm{i}y)\mathrm{d}x\mathrm{d}y\right| + C\iint \left(|f(x)| + |y||f'(x)|\right)|\chi'(y)|\,|S(x+\mathrm{i}y)|\,\mathrm{d}x\mathrm{d}y,$$

for some universal $C > 0$, where χ is a smooth cutoff function with support in $[-1, 1]$, with $\chi(y) = 1$ for $|y| \leqslant \tau/2$ and with bounded derivatives. From (6.51) all integrals can actually be restricted to a compact set. From (6.50) the second integral is $\mathrm{O}(U/N)$.

Concerning the first integral, we split it into the domains $0 < y < \eta_E$ and $\eta_E < y < 1$. By symmetry we only need to consider positive y. The integral on the domain $\{0 < y < \eta_E\}$ is easily bounded by

$$\left|\iint_{0<y<\eta_E} y f''(x)\chi(y)\Im S(x+\mathrm{i}y)\mathrm{d}x\mathrm{d}y\right| = \mathrm{O}\left(\iint_{|x-E|<\eta_E, 0<y<\eta_E} y{\eta_E}^{-2}\frac{U}{Ny}\mathrm{d}x\mathrm{d}y\right) = \mathrm{O}\left(\frac{U}{N}\right).$$

On the domain $\{\eta_E < y < 1\}$, we integrate by parts twice (first in x, then in y), and use the Cauchy-Riemann equation ($\partial_x \Im S = -\partial_y \Re S$) to obtain:

$$\iint_{y>\eta_E} y f''(x)\chi(y)\Im S(x+\mathrm{i}y)\mathrm{d}x\mathrm{d}y = -\iint_{y>\eta_E} f'(x)\partial_y(y\chi(y))\Re S(x+\mathrm{i}y)\mathrm{d}x\mathrm{d}y \tag{6.52}$$

$$-\int f'(x)\eta_E\chi(\eta_E)\Re S(x+\mathrm{i}\eta_E)\mathrm{d}x. \tag{6.53}$$

The first term (6.52) can be bounded by (6.49), it is

$$\mathrm{O}\left(\int_{\eta_E<y<\tau}\frac{U}{Ny}\mathrm{d}y\right)=\mathrm{O}\left(\frac{U(\log N)}{N}\right).$$

The second term, (6.53), can also be bounded thanks to (6.49), by $\mathrm{O}(\frac{U}{N})$, concluding the proof. □

Lemma 6.25. *Assume that rigidity at scale a holds, and moreover that the extra rigidity at scale $3a/4$ holds except for a few edge particles, in the following sense: for any $\varepsilon>0$, there are constants $c, N_0>0$ such that for any $N\geqslant N_0$ and $\hat{k}\geqslant N^{\frac{3a}{4}+\varepsilon}$ we have*

$$\mathbb{P}^{\mu}\left(|\lambda_k-\gamma_k|>N^{-\frac{2}{3}+\frac{3a}{4}+\varepsilon}(\hat{k})^{-\frac{1}{3}}\right)\leqslant e^{-N^{c}}. \tag{6.54}$$

Then for any $d>\frac{25}{28}a$ and large enough N, we have

$$\gamma_1^{(N)}\geqslant A-N^{-\frac{2}{3}+d}. \tag{6.55}$$

Proof. To prove (6.55) we will rely on lemmas 6.20 and 6.23. From the hypothesis (6.54) the conclusions (6.47) and (6.48) hold (our final choice for s,d will satisfy the required bounds: $a>d>s>a/2$). As a consequence, to check the a priori bound (6.46) uniformly on $\Omega^{(N)}(d,s,\tau)$, it is sufficient to prove that for ε small enough,

$$\begin{cases} N^{-\frac{2}{3}+\frac{3a}{4}+\varepsilon-2s}+N^{-1+\varepsilon}|z-A|^{-\frac{1}{2}}+N^{-1+\varepsilon}=\mathrm{o}(|z-A|)\\ N^{-\frac{2}{3}-4s+2a+\varepsilon}+N^{-2+a+\varepsilon}|z-A|^{-2}+N^{-1+\varepsilon}=\mathrm{o}(|z-A|)\end{cases},\quad \text{i.e., } d>\min\left\{\frac{3a}{4}-2s,2a-4s,\frac{a}{3},-\frac{1}{3}\right\}.$$

These conditions hold trivially when $s>a/2$ for example, which will be true with our choice. As $s>a/2$, the first two terms in the min are harmless, and $d>a/3$ will be satisfied in our final choice for d (we will have $d>25a/28$). The last constraint for d is trivial.

Assume that one can find arbitrarily large N such that $\gamma_1^{(N)}\leqslant A-N^{-\frac{2}{3}+d}$. We choose $z=\gamma_1^{(N)}+\mathrm{i}N^{-\frac{2}{3}+s}=E+\mathrm{i}\eta$. We have, by Lemmas 6.20 and 6.23

$$\begin{aligned}&|m_N(z)-m(z)|\leqslant C\ |z-A|^{-\frac{1}{2}}\left(\mathrm{Var}_N(z)+\frac{1}{N}|m_N'(z)|+N^{-1+\varepsilon}\right)\\ &\leqslant C\ |z-A|^{-\frac{1}{2}}\left(N^{-\frac{2}{3}+b+\varepsilon-2s}+N^{-1+\varepsilon}|z-A|^{-\frac{1}{2}}+N^{-\frac{2}{3}-4s+2a+\varepsilon}+N^{-2+a+\varepsilon}|z-A|^{-2}+N^{-1+\varepsilon}\right).\end{aligned} \tag{6.56}$$

On the other hand for any b (we will choose b greater and close to $3a/4$), we have (in the first inequality we use that the concentration scale of λ_1 around $\gamma_1^{(N)}$, $N^{-\frac{2}{3}+\frac{a}{2}}$, is much smaller than the η scale $N^{-\frac{2}{3}+s}$),

$$\begin{aligned}\frac{1}{N\eta}&\leqslant-\frac{C}{N}\mathbb{E}\left(\Im\left(\frac{1}{z-\lambda_1}\right)\right)\\ &\leqslant-\frac{C}{N}\mathbb{E}\left(\Im\left(\frac{1}{z-\lambda_1}-\frac{1}{z-\gamma_1}\right)\right)+\mathrm{O}\left(\frac{\eta}{N|z-A|^2}\right)\\ &\leqslant-\frac{C}{N}\mathbb{E}\left(\Im\left(\frac{1}{z-\lambda_1}-\frac{1}{z-\gamma_1}\right)+\sum_{2\leqslant i\leqslant N^b,\gamma_i^{(N)}\leqslant\gamma_i}\left(\Im\left(\frac{1}{z-\lambda_i}-\frac{1}{z-\gamma_i}\right)\mathbb{1}_{\lambda_i\leqslant\gamma_i}\right)\right)+\mathrm{O}\left(\frac{\eta}{N|z-A|^2}\right),\end{aligned} \tag{6.57}$$

where for the last inequality we simply used that $-\Im(1/(z-\lambda_i)-1/(z-\gamma_i))\geqslant 0$ whenever $|z-\lambda_i|\mathbb{1}_{\lambda_i\leqslant\gamma_i}\leqslant |z-\gamma_i|\mathbb{1}_{\lambda_i\leqslant\gamma_i}$. The latter inequality holds with probability $1-\mathrm{O}(e^{-N^c})$ since its complement is included in $|\lambda_i-\gamma_i^{(N)}|>|A-\gamma_1^{(N)}|=N^{-\frac{2}{3}+d}$, but λ_i is concentrated at scale $a/2$ and $d>a/2$ in our final choice for d ($d>25a/28$).

We now want to remove the assumption $\mathbb{1}_{\lambda_i\leqslant\gamma_i}$ from (6.57) and bound the associated error term. For any $i\leqslant N^b$ such that $\lambda_i>\gamma_i$ we have

$$|\lambda_i-\gamma_i|\leqslant|A-\gamma_{\lfloor N^b\rfloor}|+|\lambda_{\lfloor N^b\rfloor}-\gamma_{\lfloor N^b\rfloor}|=\mathrm{O}(N^{-\frac{2}{3}+\frac{2}{3}b}+N^{-\frac{2}{3}+\frac{3}{4}a-\frac{1}{3}b+\varepsilon})=\mathrm{O}(N^{-\frac{2}{3}+\frac{2}{3}b}) \tag{6.58}$$

where we used $b>3a/4$, and chose $\varepsilon>0$ so small that $\varepsilon\leqslant 3a/4-b$. We also used that $\lambda_{\lfloor N^b\rfloor}$ is rigid at scale $3a/4$ and these bounds hold outside of a set of exponentially small probability. Our final choice of b and d will satisfy $2b/3<d$ ($b=3a/4+\varepsilon$, $d=25a/28+\varepsilon$), consequently for any $i\leqslant N^b$ such that $\lambda_i>\gamma_i$ we have $|\lambda_i-\gamma_i|\ll N^{-\frac{2}{3}+d}$ and we can apply

$$\left|\Im\left(\frac{1}{z-\lambda}-\frac{1}{z-\gamma}\right)\right|=|\lambda-\gamma|\,\mathrm{O}\left(\frac{\eta}{|z-\gamma|^3}\right) \tag{6.59}$$

that holds for any real λ,γ, and $z=E+\mathrm{i}\eta$ such that $|\lambda-\gamma|\ll|E-\gamma|$. This condition is satisfied since

$$|\lambda_i-\gamma_i|\ll N^{-\frac{2}{3}+d}\leqslant|E-A|\leqslant|E-\gamma_i|. \tag{6.60}$$

Thus, using (6.58) and $|z-A|\leqslant|z-\gamma_i|$ we obtain

$$\mathbb{E}\left(\Im\left(\frac{1}{z-\lambda_i}-\frac{1}{z-\gamma_i}\right)\mathbb{1}_{\lambda_i>\gamma_i}\right)\leqslant\mathbb{E}(|\lambda_i-\gamma_i|\mathbb{1}_{\lambda_i>\gamma_i})\frac{\eta}{|z-A|^3}=\mathrm{O}\left(\frac{N^{-\frac{2}{3}+\frac{2}{3}b}\eta}{|z-A|^3}\right).$$

Consequently, from (6.57) we obtain

$$\frac{1}{N\eta}\leqslant-\frac{1}{N}\mathbb{E}\left(\Im\left(\frac{1}{z-\lambda_1}-\frac{1}{z-\gamma_1}\right)+\sum_{2\leqslant i\leqslant N^b,\gamma_i^{(N)}\leqslant\gamma_i}\left(\Im\left(\frac{1}{z-\lambda_i}-\frac{1}{z-\gamma_i}\right)\right)\right)+\mathrm{O}\left(\frac{\eta}{N|z-A|^2}\right)+\mathrm{O}\left(\frac{N^{-\frac{2}{3}+\frac{2}{3}b+b}\eta}{|z-A|^3}\right),$$

which implies

$$\frac{1}{N\eta}\leqslant|m_N(z)-m(z)|+\frac{1}{N}\sum_{i\geqslant N^b}\mathbb{E}\Im\left(\frac{1}{z-\lambda_i}-\frac{1}{z-\gamma_i}\right)+\frac{1}{N}\sum_{2\leqslant i\leqslant N^b,\gamma_i^{(N)}>\gamma_i}\mathbb{E}\Im\left(\frac{1}{z-\lambda_i}-\frac{1}{z-\gamma_i}\right)+\mathrm{O}\left(\frac{\eta}{N|z-A|^2}\right)+\mathrm{O}\left(\frac{N^{-\frac{2}{3}+\frac{2}{3}b+b}\eta}{|z-A|^3}\right). \tag{6.61}$$

Because of accuracy at scale $3a/4(\leqslant b)$ and concentration at scale $a/2$ for particles with index $i\geqslant N^b$, we also have (using (6.59) and (6.60))

$$\frac{1}{N}\mathbb{E}\left|\sum_{i\geqslant N^b}\Im\left(\frac{1}{z-\lambda_i}-\frac{1}{z-\gamma_i}\right)\right|\leqslant\frac{C}{N}\sum_{i\geqslant 1}\frac{N^{-\frac{2}{3}+b}i^{-\frac{1}{3}}\eta}{|z-\gamma_i|^3}\leqslant C\ N^{-\frac{1}{3}+b+s-2d}. \tag{6.62}$$

Moreover, for $\gamma_i^{(N)} \geqslant \gamma_i$ and $i \leqslant N^b$, we have for any $\varepsilon > 0$ and large enough N, $\mathbb{E}(|\lambda_i - \gamma_i|) \leqslant N^{-\frac{2}{3}+\frac{2}{3}b+\varepsilon}$, so

$$\frac{1}{N}\mathbb{E}\left|\sum_{2\leqslant i\leqslant N^b, \gamma_i^{(N)}>\gamma_i} \Im\left(\frac{1}{z-\lambda_i} - \frac{1}{z-\gamma_i}\right)\right| \leqslant \frac{1}{N}\sum_{i\leqslant N^b} \frac{N^{-\frac{2}{3}+\frac{2}{3}b+\varepsilon}\eta}{|z-\gamma_i|^3} \leqslant N^{b-1}\frac{N^{-\frac{2}{3}+\frac{2}{3}b+\varepsilon}\eta}{|z-A|^3}. \tag{6.63}$$

Consequently, when comparing the exponents of N in equations (6.61), using the estimates (6.56), (6.62) and (6.63), and using that $|z-A| \geqslant N^{-\frac{2}{3}+d}$ and $\eta = N^{-\frac{2}{3}+s}$, one of the following inequalities holds:

$$\begin{cases} -s \leqslant b - 2s - \frac{d}{2} \\ -s \leqslant -d \\ -s \leqslant 2a - 4s - \frac{d}{2} \\ -s \leqslant a - \frac{5}{2}d \\ -s \leqslant -\frac{1}{3} - \frac{d}{2} \\ -s \leqslant b + s - 2d \\ -s \leqslant \frac{5}{3}b + s - 3d. \end{cases}$$

For the choice $b = \frac{3}{4}a + \varepsilon$, $d = \frac{25}{28}a + \varepsilon$, $s = \frac{29}{56}a$, and $\varepsilon > 0$ small enough, one can check that none of these equations is satisfied (these optimal constants 25/28 and 29/56 are obtained when, for $b = 3a/4$, the third and fifth equations are equal). This is a contradiction concluding the proof. □

Proof of Proposition 6.3. For simplicity we will improve accuracy only for particles close to the edge A, $k \leqslant N/2$, the other edge being proved in a similar way. We assume rigidity at scale a. By Proposition 6.2 concentration at scale $a/2$ holds. Therefore, for any $\varepsilon > 0$, by Lemma 6.21, uniformly on $\Sigma_{\rm Int}^{(N)}(3a/4+\varepsilon,\tau)$ we have

$$\mathrm{Var}_N(z) \ll \frac{N^{\frac{3a}{4}}}{N\eta}\max(\kappa_E^{\frac{1}{2}}, \eta^{\frac{1}{2}}).$$

This easily implies that

$$\mathrm{Var}_N(z) \ll \max(\kappa_E, \eta) \tag{6.64}$$

uniformly on $\Sigma_{\rm Int}^{(N)}(3a/4+\varepsilon,\tau)$. To see this, as $\eta \geqslant N^{-1+\frac{3a}{4}}\kappa_E^{-1/2}$ we always have $\frac{N^{\frac{3a}{4}}}{N\eta}\kappa_E^{\frac{1}{2}} \leqslant \kappa_E$. Moreover, if $\eta \geqslant \kappa_E$ we have $\eta \geqslant N^{-1+\frac{3a}{4}}\kappa_E^{-\frac{1}{2}} \geqslant N^{-1+\frac{3a}{4}}\eta^{-\frac{1}{2}}$, so $\eta \geqslant N^{-\frac{2}{3}+\frac{a}{2}}$, so $\frac{N^{\frac{3a}{4}}}{N\eta}\eta^{\frac{1}{2}} \leqslant \eta$, completing the proof of (6.64).

Consequently, the conclusion of Lemma 6.19 holds: uniformly on $\Sigma^{(N)}(3a/4+\varepsilon,\tau)$, we have

$$|m_N(z) - m(z)| \leqslant c\left(\frac{N^\varepsilon}{N\eta} + |\,\mathrm{Var}_N(z)|\frac{1}{\max(\sqrt{\kappa_E}, \sqrt{\eta})}\right) \leqslant c\frac{N^{\frac{3a}{4}}}{N\eta}.$$

One can therefore apply Lemma 6.24 with the choice $\widetilde{\varrho} = \varrho_1^{(N)} - \varrho$, $\eta = N^{-1+\frac{3a}{4}}$ and $U = N^{\frac{3a}{4}}$ (the extra assumption (6.50) about the macroscopic behaviour of $m_N - m$ holds thanks to Lemma 6.6 and condition (6.51) is satisfied thanks to (6.7)): we proved that, for any $b > 3a/4$, we have

$$\Big|\int f(\lambda)(\varrho_1^{(N)}(\lambda) - \varrho(\lambda))\mathrm{d}\lambda\Big| \leqslant C\ N^{-1+b}, \tag{6.65}$$

for any $E \in (A, B)$. Here $f = f_E = f_{E,\eta_E}$ as defined in Lemma 6.24. We choose some $E \geqslant A + N^{-\frac{2}{3}+\frac{a}{2}}$, so that $\eta_E = \kappa_E^{-1/2}\eta \leqslant N^{-\frac{2}{3}+\frac{a}{2}}$, thus $E - \eta_E \geqslant A$. We therefore have, using (6.65),

$$\int_{-\infty}^{E} \varrho_1^{(N)} \geqslant \int f_{E-\eta_E}\varrho_1^{(N)} = \int (\varrho_1^{(N)} - \varrho) f_{E-\eta_E} + \int \varrho f_{E-\eta_E} = \mathrm{O}\left(N^{-1+b}\right) + \int_{-\infty}^{E} \varrho + \mathrm{O}(\eta),$$
$$\int_{-\infty}^{E} \varrho_1^{(N)} \leqslant \int f_E\varrho_1^{(N)} = \int (\varrho_1^{(N)} - \varrho) f_E + \int \varrho f_E = \mathrm{O}\left(N^{-1+b}\right) + \int_{-\infty}^{E} \varrho + \mathrm{O}(\eta).$$

The error $O(\eta)$ can be included into the first error term. We first assume that $k \geqslant N^b$, and we choose $E = \gamma_k^{(N)}$ (as defined in (6.1)) in the above equations, where the condition $E - \eta_E \geqslant A$ is satisfied when $k \geqslant N^b$. We get $|\int_{\gamma_k^{(N)}}^{\gamma_k} \varrho| = \mathrm{O}\left(N^{-1+b}\right)$, hence

$$\left|\left(\gamma_k^{(N)}\right)^{3/2} - (\gamma_k)^{3/2}\right| = \mathrm{O}(N^{-1+b}). \tag{6.66}$$

This implies accuracy at scale $3a/4$: if $k \geqslant N^b$ we have $\gamma_k^{3/2} \geqslant cN^{-1+b}$, so by linearizing (6.66) we obtain

$$\gamma_k^{(N)} = (\gamma_k^{3/2} + \mathrm{O}(N^{-1+b}))^{2/3} = \gamma_k\left(1 + \frac{N^{-1+b}}{\gamma_k^{3/2}}\right)^{2/3} = \gamma_k + \mathrm{O}\left(N^{-1+b}/\gamma_k^{1/2}\right) = \gamma_k + \mathrm{O}(N^{-\frac{2}{3}+b}k^{-\frac{1}{3}}).$$

We proved that accuracy at scale $3a/4$ holds provided that $\hat{k} \geqslant N^b$, for b arbitrarily close to $3a/4$. We know that, for such k, together with concentration at scale $a/2$ this implies rigidity at scale $3a/4$ (by the same reasoning as in the proof of Theorem 2.4). This allows us first to use Lemma 6.25 to obtain that, for any $\varepsilon > 0$, for large enough N we have

$$\gamma_1^{(N)} \geqslant A - N^{-\frac{2}{3}+\frac{25}{28}a}. \tag{6.67}$$

It also allows us to use Lemmas 6.22 and 6.19 together to conclude that for any $d > 2a/3$ and $\tau > 0$ small enough, we have, uniformly in $\Sigma_{\mathrm{Ext}}^{(N)}(d, \tau)$,

$$|m_N(z) - m(z)| \leqslant \frac{1}{N\eta}.$$

By part $b)$ of Lemma 6.24, with $\eta = N^{-1+\frac{3d}{2}}$, $E = A - 2N^{-\frac{2}{3}+d}$, this implies that there is a function $f = 1$ on $(-\infty, A - 2N^{-\frac{2}{3}+d}]$, $f = 0$ on $[-N^{-\frac{2}{3}+d}, \infty)$, such that

$$\left|\int f(\lambda)\varrho_1^{(N)}(\lambda)\mathrm{d}\lambda\right| \leqslant C\ \frac{\log N}{N},$$

(since in this interval $\varrho = 0$), hence there is some $c > 0$ such that for large enough N we have

$$\gamma_{\lfloor c\log N\rfloor}^{(N)} > A - N^{-\frac{2}{3}+d}.$$

In particular, as $N^{-\frac{2}{3}+\frac{11}{12}a}j^{-\frac{1}{3}} = N^{-\frac{2}{3}+\frac{2}{3}a}$ when $j = N^{\frac{3}{4}a}$, the previous equation proves accuracy at scale $11a/12$ for any λ_i with $i \in [\![C\log N, N^{\frac{3a}{4}+\varepsilon}]\!]$. For the remaining $i \in [\![1, C\log N]\!]$, we use (6.67), which also gives accuracy at scale $11a/12$ because $25/28 < 11/12$. □

6.5 Proof of Theorem 3.1

This proof goes along the same lines as the one of Theorem 2.4 up to two major differences that make it easier:

- For large enough N, the Hamitonian $\mathcal{H}_{\mathbf{y}}$ will be shown to be convex, so there is no need for introducing any convexified measure.
- In the hypothesis of Theorem 3.1 about rigidity for local measures, the definition of the good set $\mathcal{R}_K^*(\xi)$ already assumes a strong form of accuracy: $|\mathbb{E}^{\sigma_{\mathbf{y}}}(x_k)-\gamma_k|\leqslant N^{-\frac{2}{3}+\xi}k^{-\frac{1}{3}}$. Therefore there will be no need to prove an analogue of Proposition 6.3.

By the following easy lemma, the first particle x_1 satisfies a strong form of rigidity concerning deviations on the left.

Lemma 6.26. *There exists a constants $c, C>0$ depending only on β, V, ξ such that for any K and $\mathbf{y}\in\mathcal{R}^*=\mathcal{R}_K^*(\xi)$ we have, for any $u>0$,*

$$\mathbb{P}^{\sigma_{\mathbf{y}}}\left(x_1\leqslant -uN^{-\frac{2}{3}+\xi}\right)\leqslant C\,e^{-cu^2}.$$

Proof. We note

$$\mathcal{H}_{\mathbf{y}}(\mathbf{x})=\frac{1}{2}\sum_I V_{\mathbf{y}}(x_i)-\frac{1}{N}\sum_{i<j}\log(x_j-x_i).$$

Then

$$Z_{\hat{\sigma}_{\mathbf{y}}}\leqslant 2Z_{\hat{\sigma}_{\mathbf{y}}}\frac{1}{Z_{\hat{\sigma}_{\mathbf{y}}}}\int e^{-\beta N\left(\mathcal{H}_{\mathbf{y}}(x)+\frac{1}{N}\sum_I\Theta(N^{\frac{2}{3}-\xi}x_i)+\frac{1}{N}\sum_I\Theta(N^{\frac{2}{3}-\xi}x_i)\right)}\mathrm{d}x=2Z_{\sigma_{\mathbf{y}}}.$$

In the above inequality we used $\mathbb{P}^{\hat{\sigma}_{\mathbf{y}}}(x_1\geqslant -N^{-\frac{2}{3}+\xi})\geqslant 1/2$ (because $\mathbf{y}\in\mathcal{R}^*$) and $\Theta(N^{\frac{2}{3}-\xi}x_i)=0$ when $x_i\geqslant -N^{-\frac{2}{3}+\xi}$. We then easily get, for $u\geqslant 2$,

$$\begin{aligned}\mathbb{P}^{\sigma_{\mathbf{y}}}(x_1\leqslant -uN^{-\frac{2}{3}+\xi})=\frac{Z_{\hat{\sigma}_{\mathbf{y}}}}{Z_{\sigma_{\mathbf{y}}}}\frac{1}{Z_{\hat{\sigma}_{\mathbf{y}}}}\int e^{-\beta N\left(\hat{\mathcal{H}}^{\sigma}_{\mathbf{y}}+\frac{1}{N}\sum_I\Theta(N^{\frac{2}{3}-\xi}x_i)\right)}\mathbb{1}\left(x_1\leqslant -uN^{-\frac{2}{3}+\xi}\right)\mathrm{d}x\\ \leqslant 2\mathbb{E}^{\hat{\sigma}_{\mathbf{y}}}\left(e^{-\beta\Theta(N^{\frac{2}{3}-\xi}x_1)}\mathbb{1}\left(x_1\leqslant -uN^{-\frac{2}{3}+\xi}\right)\right)\leqslant 2e^{-\beta(u-1)^2}.\end{aligned}$$

This concludes the proof (bounding the probability by 1 when $0<u<2$). □

The following notion of conditional rigidity at scale M will be useful in our proof of optimal conditional rigidity, i.e., Theorem 3.1. It is analogous to Definition 6.1 in [25], which was in the context of bulk eigenvalues.

Definition 6.27. *Given $\xi>0$, we will say that the measure $\sigma_{\mathbf{y}}$ satisfies conditional rigidity at scale M if there exists $c>0$ such that for large enough N we have, for any $\mathbf{y}\in\mathcal{R}_K^*(\xi)$, $\ell\in I$ and $u>0$,*

$$\mathbb{P}^{\sigma_{\mathbf{y}}}\left(|x_\ell-\gamma_\ell|>N^{-\frac{2}{3}+\xi}\ell^{-\frac{1}{3}}u+N^{-\frac{2}{3}}\ell^{-\frac{1}{3}}M\right)\leqslant e^{-cu^2}.$$

The parameter ξ is considered fixed in this definition.

Following ideas from Section 6.1 in [25], we set $\eta = \xi/3$ and will consider a sequence $N^\xi = M_1 < \cdots < M_A = CKN^{-2\eta}$ (for some large constant C) such that for any $j \in [\![1, A-1]\!]$ we have $M_{j+1}/M_j \sim N^\eta$ (meaning that $cN^\eta < M_{j+1}/M_j < CN^\eta$). Here A is a constant bounded by $\mathrm{O}(\xi^{-1})$. Our first task is to prove conditional rigidity at scale M_A for $\sigma_{\mathbf{y}}$.

Step 1: conditional rigidity at a large scale. The Hamiltonian $\mathcal{H}_{\mathbf{y}}^\sigma$ satisfies the following convexity bound: for any $\mathbf{v} \in \mathbb{R}^K$,

$$\langle \mathbf{v}, (\nabla^2 \mathcal{H}_{\mathbf{y}}^\sigma)\mathbf{v}\rangle \geqslant c \sum_{i\in I}\left(\sum_{j\notin I}\frac{1}{N(x_i-y_j)^2} - 2W + N^{-1+\frac{4}{3}-2\xi}\Theta''(N^{\frac{2}{3}-\xi}x_i)\right)|v_i|^2.$$

If $x_i \geqslant -N^{-\frac{2}{3}+\xi}$, we get that $\sum_{j\notin I}\frac{1}{N(x_i-y_j)^2} \geqslant cN^{-1}\sum_{j=K}^{N/2}(j/N)^{-4/3} \geqslant c(N/K)^{1/3}$ (remember that the rigidity exponent ξ is much smaller than the exponent δ in (3.1), so $|x_i - y_j| \leqslant CN^{-\frac{2}{3}+\xi} + |y_j| \leqslant CN^{-\frac{2}{3}+\xi} + C(j/N)^{\frac{2}{3}} + CN^{-\frac{2}{3}+\xi}j^{-\frac{1}{3}} \leqslant C(j/N)^{\frac{2}{3}}$ for $j \geqslant K$). If $x_i \leqslant -N^{-\frac{2}{3}+\xi}$ then $N^{-1+\frac{4}{3}-2\xi}\Theta''(N^{\frac{2}{3}-\xi}x_i) \geqslant cN^{1/3-2\xi} \geqslant c(N/K)^{\frac{1}{3}}$, so in all cases we proved the inequality

$$\nabla^2\mathcal{H}_{\mathbf{y}}^\sigma \geqslant c\,(N/K)^{\frac{1}{3}}.$$

The measure $\sigma_{\mathbf{y}}$ therefore satisfies a logarithmic Sobolev inequality with constant of order $K^{-1/3}N^{4/3}$, so we have for any $\ell \in [\![1, K-M_A]\!]$ that

$$\mathbb{P}^{\sigma_{\mathbf{y}}}\left(|x_\ell^{[M_A]} - \mathbb{E}^{\sigma_{\mathbf{y}}}(x_\ell^{[M_A]})| > v\right) \leqslant \exp(-cM_AK^{-1/3}N^{4/3}v^2),\ v \geqslant 0.$$

In particular,

$$\mathbb{P}^{\sigma_{\mathbf{y}}}\left(|x_\ell^{[M_A]} - \mathbb{E}^{\sigma_{\mathbf{y}}}(x_\ell^{[M_A]})| > N^{-2/3+\xi}\ell^{-1/3}u\right) \leqslant e^{-cu^2}, \tag{6.68}$$

because $N^{2\xi} > K^{2\eta}$. Moreover, using the definition (3.5), we know that

$$\left|\mathbb{E}^{\sigma_{\mathbf{y}}}(x_\ell^{[M_A]}) - \gamma_\ell^{[M_A]}\right| \leqslant CN^{-\frac{2}{3}+\xi}\ell^{-1/3}.$$

We therefore proved that

$$\mathbb{P}^{\sigma_{\mathbf{y}}}\left(|x_\ell^{[M_A]} - \gamma_\ell^{[M_A]}| > CN^{-\frac{2}{3}+\xi}\ell^{-\frac{1}{3}} + N^{-2/3+\xi}\ell^{-1/3}u\right) \leqslant e^{-cu^2}.$$

Moreover, from easy ordering considerations we have for any $\ell \in [\![M_A, K-M_A]\!]$

$$\begin{aligned} x_\ell - \gamma_\ell &\leqslant (x_\ell^{[M_A]} - \gamma_\ell^{[M_A]}) + (\gamma_\ell^{[M_A]} - \gamma_\ell) \leqslant (x_\ell^{[M_A]} - \gamma_\ell^{[M_A]}) + CM_AN^{-\frac{2}{3}}\ell^{-\frac{1}{3}}, \\ x_\ell - \gamma_\ell &\geqslant (x_{\ell-M_A}^{[M_A]} - \gamma_{\ell-M_A}^{[M_A]}) + (\gamma_{\ell-M_A}^{[M_A]} - \gamma_\ell) \geqslant (x_{\ell-M_A}^{[M_A]} - \gamma_{\ell-M_A}^{[M_A]}) - CM_AN^{-\frac{2}{3}}\ell^{-\frac{1}{3}}. \end{aligned} \tag{6.69}$$

Thus

$$\mathbb{P}^{\sigma_{\mathbf{y}}}\left(|x_\ell - \gamma_\ell| > CN^{-\frac{2}{3}+\xi}\ell^{-\frac{1}{3}}M_A + N^{-2/3+\xi}\ell^{-1/3}u\right) \leqslant e^{-cu^2}. \tag{6.70}$$

If $\ell \in [\![1, M_A]\!]$, then the bound (6.69) still holds, and for concentration on the left we simply use $x_\ell - \gamma_\ell \geqslant x_1 - C\left(\ell N^{-1}\right)^{\frac{2}{3}}$, which yields

$$|x_\ell - \gamma_\ell| \leqslant |\min(x_1, 0)| + |x_\ell^{[M_A]} - \gamma_\ell^{[M_A]}| + CM_AN^{-\frac{2}{3}}\ell^{-\frac{1}{3}}.$$

Using Lemma 6.26 and (6.68), this inequality proves that the desired rigidity (6.70) also holds for $\ell \in [\![1, M_A]\!]$. The case $\ell \in [\![K - M_A, K]\!]$ is more elementary, the boundary on the right being fixed: $|x_\ell - \gamma_\ell| \leqslant |y_{K+1} - \gamma_{K+1}| + |\gamma_{K+1} - \gamma_\ell| + |x^{[M_A]}_{\ell-M_A} - \gamma^{[M_A]}_{\ell-M_A}| + |\gamma_\ell - \gamma^{[M_A]}_{\ell-M_A}|$, and the desired result (6.70) follows from the definition of $\mathcal{R}$ to bound $|y_{K+1} - \gamma_{K+1}|$, (6.68) and bounding of $|\gamma_{K+1} - \gamma_\ell|$ and $|\gamma_\ell - \gamma^{[M_A]}_{\ell-M_A}|$ by $CM_A N^{-\frac{2}{3}} \ell^{-\frac{1}{3}}$. This concludes the proof of conditional rigidity at scale M_A for $\sigma_{\mathbf{y}}$.

Step 2: induction on the scales. In order to consider smaller scales, we will need in the following version of the locally constrained measure (6.34): for any $\ell \in [\![1, K - M]\!]$, define the probability measure

$$\begin{aligned} \mathrm{d}\omega_{\mathbf{y}}{}^{(\ell,M)}(\mathbf{x}) &\sim \exp\left(-\beta \phi^{(\ell,M)}_{\mathrm{loc}}(\mathbf{x})\right) \mathrm{d}\sigma_{\mathbf{y}}(\mathbf{x}), \\ \phi^{(\ell,M)}_{\mathrm{loc}}(\mathbf{x}) &= \sum_{i<j, i,j \in I^{(\ell,M)}} \theta\left(\frac{N^{\frac{2}{3}} \ell^{\frac{1}{3}}}{M N^{2\eta}} (x_i - x_j)\right), \end{aligned} \tag{6.71}$$

where $\theta(x) = (x-1)^2 \mathbb{1}_{x>1} + (x+1)^2 \mathbb{1}_{x<-1}$ and $I^{(\ell,M)} = [\![\ell, \ell + M - 1]\!]$. Note that our definition of $\phi^{(\ell,M)}_{\mathrm{loc}}$ only differs from $\phi^{(\ell,M)}$ (see Definition 6.11) concerning the extra factor $N^{2\eta}$. We now present our induction on the scales: we will show that if the following three conditions hold for the index j then they are also true for $j - 1$.

(i) There exists $c > 0$ such that for large enough N, for any $\ell \in [\![1, K - M_j]\!]$ and $u > 0$

$$\mathbb{P}^{\sigma_{\mathbf{y}}}\left(|x^{[M_j]}_\ell - \gamma^{[M_j]}_\ell| > N^{-\frac{2}{3}+\xi} \ell^{-\frac{1}{3}} u\right) \leqslant e^{-cu^2}.$$

(ii) The following conditional rigidity at scale M_j holds: there exists $c, C > 0$ such that for large enough N, for any $\ell \in I$ and $u > 0$, we have

$$\mathbb{P}^{\sigma_{\mathbf{y}}}\left(|x_\ell - \gamma_\ell| > N^{-\frac{2}{3}+\xi} \ell^{-\frac{1}{3}} u + C\, N^{-\frac{2}{3}} \ell^{-\frac{1}{3}} M_j\right) \leqslant e^{-cu^2}.$$

(iii) The following entropy bound holds, for large enough N and any $\ell \in [\![1, K - M_j]\!]$:

$$\mathrm{S}(\sigma_{\mathbf{y}} \mid \omega^{(\ell,M_j)}_{\mathbf{y}}) \leqslant e^{-cM_j^2 N^{-2\eta}}.$$

The initial step $j = A$ of the induction was just checked in Step1, concerning points (i) and (ii) (see equations (6.68) and (6.70)). Concerning (iii), it follows easily from (ii): if $\phi^{(\ell,M_A)}_{\mathrm{loc}}(\mathbf{x}) > 0$ then $x_1 < -N^{-\frac{2}{3}} M_A N^{2\eta}$, which has $\sigma_{\mathbf{y}}$-probability bounded by $\exp(-cM_A^2 N^{-2\eta})$ (by (ii)). The logarithmic Sobolev inequality for $\sigma_{\mathbf{y}}$ therefore allows us to conclude:

$$\mathrm{S}(\sigma_{\mathbf{y}} \mid \omega^{(\ell,M_A)}_{\mathbf{y}}) \leqslant CN^C \mathbb{E}^{\sigma_{\mathbf{y}}} |\nabla \phi^{(\ell,M_A)}_{\mathrm{loc}}|^2 \leqslant CN^C \exp(-cM_A^2 N^{-2\eta}) \leqslant \exp(-c' M_A^2 N^{-2\eta}). \tag{6.72}$$

We now prove that (i),(ii),(iii) with M_j implies the same result with M_{j-1}. That (i) implies (ii) is easy and follows from the exact same argument allowing to conclude about the initial local rigidity (6.70). To prove (iii) from (ii), note that if $\phi^{(\ell,M_j)}_{\mathrm{loc}}(\mathbf{x}) > 0$ then for some $i \in I^{(\ell,M_j)}$ we have $|x_i - \gamma_i| > cN^{-\frac{2}{3}} i^{-\frac{1}{3}} M_j N^{2\eta}$. From (ii) this has probability (for $\sigma_{\mathbf{y}}$) bounded by $e^{-cM_j^2 N^{-2\eta}}$. One then concludes similarly to (6.72). We therefore now only need to prove (i) at scale M_{j-1}. We have the following analogue of equation (6.12) in [25]:

for any choice $\ell_j \in [\![1, K - M_j]\!]$ and $\ell_{j-1} \in [\![1, K - M_{j-1}]\!]$ such that $[\![\ell_{j-1}, \ell_{j-1} + M_{j-1}]\!] \subset [\![\ell_j, \ell_j + M_j]\!]$ we have

$$\mathbb{P}^{\omega_{\mathbf{y}}^{(\ell,M_j)}}\left(\left|x_{\ell_{j-1}}^{[M_{j-1}]} - x_{\ell_j}^{[M_j]} - \mathbb{E}^{\omega_{\mathbf{y}}^{(\ell,M_j)}}\left(x_{\ell_{j-1}}^{[M_{j-1}]} - x_{\ell_j}^{[M_j]}\right)\right| > N^{-\frac{2}{3}}\ell_j^{-\frac{1}{3}}N^{5\eta/2}u\right) \leqslant e^{-cu^2}. \tag{6.73}$$

The proof of the above equation relies on Herbst's argument for concentration of measure from the logarithmic Sobolev inequality, and Lemma 3.9 in [8] to obtain a local LSI. Note that the assumptions of this Lemma are satisfied in our case: one can decompose $\mathcal{H}_{\mathbf{y}}^{\sigma} = \mathcal{H}_1 + \mathcal{H}_2$ where

$$\mathcal{H}_1(\mathbf{x}) = \frac{1}{N}\phi_{\text{loc}}^{(\ell,M_j)}(\mathbf{x}) - \frac{1}{N}\sum_{s<t,s,t\in I^{\ell,M}} \log|x_s - x_t|$$

and $\mathcal{H}_2$ is convex, thanks to the confining term Θ which applies to all x_i's, $i \in I$. Compared to (6.12) in [25], we obtained $N^{5\eta/2}$ instead of $K^{5\eta/2}$ due to $\sqrt{M_j/M_{j-1}} = N^{\eta/2}$ and the factor $N^{2\eta}$ in (6.71) instead of $K^{2\eta}$.

Moreover, using the boundedness of the x_k's on the right and Lemma 6.26 on the left, similarly to (6.72) we easily obtain

$$\left|\mathbb{E}^{\omega_{\mathbf{y}}^{(\ell,M_j)}}(x_i) - \mathbb{E}^{\sigma_{\mathbf{y}}}(x_i)\right| \leqslant C\sqrt{\mathrm{S}(\sigma_{\mathbf{y}} \mid \omega_{\mathbf{y}}^{(\ell,M_j)})} \leqslant \exp(-cM_j^2 N^{-2\eta}).$$

We know from (3.5) that $|\mathbb{E}^{\sigma_{\mathbf{y}}} x_i - \gamma_i| \leqslant N^{-\frac{2}{3}+\xi} i^{-\frac{1}{3}}$, so

$$\left|\mathbb{E}^{\omega_{\mathbf{y}}^{(\ell,M_j)}}(x_i) - \gamma_i\right| \leqslant CN^{-\frac{2}{3}+\xi} i^{-\frac{1}{3}}.$$

Changing $\omega_{\mathbf{y}}^{(\ell,M_j)}$ into $\sigma_{\mathbf{y}}$ in the equation (6.73) implies an error of order $\sqrt{\mathrm{S}(\sigma_{\mathbf{y}} \mid \omega_{\mathbf{y}}^{(\ell,M_j)})}$, which yields

$$\mathbb{P}^{\sigma_{\mathbf{y}}}\left(\left|x_{\ell_{j-1}}^{[M_{j-1}]} - x_{\ell_j}^{[M_j]} - \left(\gamma_{\ell_{j-1}}^{[M_{j-1}]} - \gamma_{\ell_j}^{[M_j]}\right)\right| > N^{-\frac{2}{3}}\ell_j^{-\frac{1}{3}}N^{5\eta/2}u + N^{-\frac{2}{3}+\xi}\ell_j^{-\frac{1}{3}}\right) \leqslant \exp(-cu^2) + \exp(-cM_j^2 N^{-2\eta}).$$

Combining this with (i) and using $\xi = 3\eta$ we get

$$\mathbb{P}^{\sigma_{\mathbf{y}}}\left(\left|x_{\ell_{j-1}}^{[M_{j-1}]} - \gamma_{\ell_{j-1}}^{[M_{j-1}]}\right| > N^{-\frac{2}{3}+\xi}\ell_j^{-\frac{1}{3}}u + N^{-\frac{2}{3}+\xi}\ell_j^{-\frac{1}{3}}\right) \leqslant \exp(-cu^2) + \exp(-cM_j^2 N^{-2\eta}).$$

If $\ell_{j-1} \leqslant K - M_j$ we can choose $\ell_j = \ell_{j-1}$ in the above equation. If $\ell_{j-1} \in [\![K - M_j, K]\!]$, then we choose $\ell_j = \lfloor K - M_j \rfloor$ and we have $\ell_j \sim \ell_{j-1}$: in any case we therefore proved

$$\mathbb{P}^{\sigma_{\mathbf{y}}}\left(\left|x_{\ell_{j-1}}^{[M_{j-1}]} - \gamma_{\ell_{j-1}}^{[M_{j-1}]}\right| > CN^{-\frac{2}{3}+\xi}\ell_{j-1}^{-\frac{1}{3}}u + CN^{-\frac{2}{3}+\xi}\ell_{j-1}^{-\frac{1}{3}}\right) \leqslant \exp(-cu^2) + \exp(-cM_j^2 N^{-3\eta}).$$

We therefore proved (i) on scale M_{j-1} provided that $u \leqslant cM_j N^{-\eta}$.

We now assume $u \geqslant cM_j N^{-\eta}$. Note that $N^{-\frac{2}{3}+\xi}\ell_{j-1}^{-\frac{1}{3}}u \geqslant c(N^{-\frac{2}{3}}\ell_{j-1}^{-\frac{1}{3}}M_j + N^{-\frac{2}{3}+\xi}\ell_{j-1}^{-\frac{1}{3}}u)$, which allows the following bounds thanks to (ii):

$$\begin{aligned}\mathbb{P}^{\sigma_{\mathbf{y}}}\left(\left|x_{\ell_{j-1}}^{[M_{j-1}]} - \gamma_{\ell_{j-1}}^{[M_{j-1}]}\right| > N^{-\frac{2}{3}+\xi}\ell_{j-1}^{-\frac{1}{3}}u\right) &\leqslant \mathbb{P}^{\sigma_{\mathbf{y}}}\left(\left|x_{\ell_{j-1}}^{[M_{j-1}]} - \gamma_{\ell_{j-1}}^{[M_{j-1}]}\right| > c(N^{-\frac{2}{3}}\ell_{j-1}^{-\frac{1}{3}}M_j + N^{-\frac{2}{3}+\xi}\ell_{j-1}^{-\frac{1}{3}}u)\right)\\ &\leqslant \sum_{\ell\in I^{(\ell_{j-1},M_{j-1})}} \mathbb{P}^{\sigma_{\mathbf{y}}}\left(|x_\ell - \gamma_\ell| > c(N^{-\frac{2}{3}}\ell_{j-1}^{-\frac{1}{3}}M_j + N^{-\frac{2}{3}+\xi}\ell_{j-1}^{-\frac{1}{3}}u)\right) \leqslant M_{j-1}e^{-cu^2} \leqslant e^{-c'u^2}.\end{aligned}$$

This concludes the induction. Notice that the constant c in the Gaussian tail $\exp(cu^2)$ deteriorates at each step, but we perform only finitely many steps. The result (ii) at the final scale $M_1 = N^{\xi}$ finishes the proof of Theorem 3.1.

7 Analysis of the local Gibbs measure

Before studying $\sigma_{\mathbf{y}}$, we remind well-known properties of the equilibrium density, at the macroscopic level: $\varrho = \varrho_V$ can be obtained as the unique solution to the variational problem

$$\inf\Big\{\int_{\mathbb{R}} V(t)\mathrm{d}\varrho(t) - \int_{\mathbb{R}}\int_{\mathbb{R}} \log|t-s|\mathrm{d}\varrho(t)\mathrm{d}\varrho(s)\ :\ \varrho \text{ is a probability measure}\Big\}, \tag{7.1}$$

and it satisfies the following equation

$$\frac{1}{2}V'(x) = \int \frac{\varrho(y)\mathrm{d}y}{x-y}, \qquad x \in [A,B]. \tag{7.2}$$

7.1 Rescaling

We now switch to the microscopic coordinates with a scaling adapted to the left edge of the spectrum at $A = 0$, i.e., we consider the scaling transformation $\lambda_j \to 3/2\, N^{2/3}\lambda_j$. In this new coordinate, the gaps of the points at the edge are order one and the gaps in the bulk are of order $N^{-1/3}$. With a slight abuse of notation we will still use the same letters x_j, y_j for the internal and external points, but *from now on they should be understood in the microscopic coordinates except in the Appendix A.* This means that the classical location of the k-th point and the k-th gap are

$$\gamma_k = (\widehat{k})^{2/3}\big(1 + O((k/N)^{2/3})\big), \qquad \gamma_{k+1} - \gamma_k \sim (\widehat{k})^{-1/3}, \tag{7.3}$$

for any $k \in [\![1, N]\!]$, see (2.10) (here the constant is adjusted to be 1, from the choice of normalization (2.8) and the scaling $\lambda_j \to 3/2\, N^{2/3}\lambda_j$). Recall we partition the external and internal points as

$$(\mathbf{x}, \mathbf{y}) = (x_1, x_2, \dots, x_K, y_{K+1}, y_{K+2}, \dots, y_N).$$

Given a boundary condition $\mathbf{y}$, we again set $J_{\mathbf{y}} = (-\infty, y_{K+1}) =: (y_-, y_+)$ to be the configuration interval, and let $\alpha_j = \alpha_j(\mathbf{y})$ be $(K+1)$-quantiles of the density in $J_{\mathbf{y}}$ exactly as in (3.10):

$$\int_0^{\alpha_j} \varrho(s)\mathrm{d}s = \frac{j}{K+1}\int_0^{y_+} \varrho(s)\mathrm{d}s, \qquad j \in I. \tag{7.4}$$

The measure σ from (3.2) in microscopic coordinate reads as

$$\sigma(\mathrm{d}\mathbf{x}) := \frac{Z}{Z_\sigma} e^{-2\beta\sum_{i\in I}\Theta(N^{-\xi}x_i)}\mu(\mathrm{d}\mathbf{x}), \tag{7.5}$$

and the local measures $\sigma_{\mathbf{y}}$ are defined analogously:

$$\sigma_{\mathbf{y}}(\mathrm{d}\mathbf{x}) = \frac{1}{Z_{\mathbf{y},\sigma}} e^{-\beta N \mathcal{H}^\sigma_{\mathbf{y}}(\mathbf{x})}\mathrm{d}\mathbf{x},$$

with Hamiltonian

$$\begin{aligned}\mathcal{H}^\sigma_{\mathbf{y}}(\mathbf{x}) :=& \frac{2}{N}\sum_{i\in I}\Theta(N^{-\xi}x_i) + \sum_{i\in I}\frac{1}{2}V_{\mathbf{y}}(x_i) - \frac{1}{N}\sum_{\substack{i,j\in I\\ i<j}}\log|x_j - x_i|,\\ V_{\mathbf{y}}(x) :=& V(xN^{-2/3}) - \frac{2}{N}\sum_{j\notin I}\log|x - y_j|.\end{aligned} \tag{7.6}$$

Here $V_{\mathbf{y}}(x)$ can be viewed as the external potential of the log-gas.

Recall the rigidity bound (4.2) for σ. The definitions of the good boundary conditions (3.4), (3.5) and (3.6) are also rescaled:

$$\begin{aligned}\mathcal{R} = \mathcal{R}_K(\xi) &:= \{\mathbf{y} \,:\, |y_k - \gamma_k| \leqslant N^{\xi}(\widehat{k})^{-1/3},\ k \notin I\},\\ \mathcal{R}^* = \mathcal{R}^*_K(\xi) &:= \{\mathbf{y} \in \mathcal{R}_K(\xi) \,:\, |\mathbb{E}^{\sigma_{\mathbf{y}}} x_k - \gamma_k| \leqslant N^{\xi}(\widehat{k})^{-\frac{1}{3}},\ \mathbb{P}^{\widehat{\sigma}}(x_1 \geqslant \gamma_1 - N^{\xi}) \geqslant 1/2\ \ \forall k \in I\},\\ \mathcal{R}^{\#} = \mathcal{R}^{\#}_K(\xi) &:= \{\mathbf{y} \in \mathcal{R}_K(\xi/3) \,:\, |y_{K+1} - y_{K+2}| \geqslant N^{-\xi}K^{-1/3}\}.\end{aligned}$$

In the new coordinates, the lower bound (5.10) on the Hessian of $\mathcal{H}^{\sigma}_{\mathbf{y}}$ reads as

$$(\mathcal{H}^{\sigma}_{\mathbf{y}})'' \geqslant cK^{-1/3}N^{-1}, \qquad \mathbf{y} \in \mathcal{R}. \tag{7.7}$$

We also have the rescaled form of (3.14) that for any $\mathbf{y} \in \mathcal{R}$

$$|\alpha_j - \gamma_j| \leqslant CN^{\xi}\,\frac{j^{2/3}}{K} \leqslant CN^{\xi}\,j^{-1/3}, \qquad j \in I. \tag{7.8}$$

The main universality result on the local measures is the following theorem, which is essentially the rescaled version of Theorem 3.3. We will first complete the proof of Theorem 3.3, then the rest of the paper is devoted to the proof of Theorem 7.1 which will be completed at the end of Section 10.4.

Theorem 7.1 (Edge universality for local measures)**.** *We assume the conditions of Theorem 3.3, in particular that the parameters* ξ, δ, ζ *and* K *satisfy* (3.11) *and* (3.12)*. Let* $\mathbf{y} \in \mathcal{R}^{\#}_{K,V,\beta}(\xi) \cap \mathcal{R}^*_{K,V,\beta}(\xi)$ *and* $\widetilde{\mathbf{y}} \in \mathcal{R}^{\#}_{K,\widetilde{V},\beta}(\xi) \cap \mathcal{R}^*_{K,\widetilde{V},\beta}(\xi)$ *be two different boundary conditions satisfying*

$$y_{K+1} = \widetilde{y}_{K+1}. \tag{7.9}$$

In particular, we know that

$$|\mathbb{E}^{\sigma_{\mathbf{y}}} x_j - \alpha_j| + |\mathbb{E}^{\widetilde{\sigma}_{\widetilde{\mathbf{y}}}} x_j - \widetilde{\alpha}_j| \leqslant CN^{\xi} j^{-1/3}, \qquad j \in I, \tag{7.10}$$

and

$$\mathbb{P}^{\hat{\sigma}}(x_1 \geqslant \gamma_1 - N^{\xi}/2) \geqslant 1/2, \qquad \mathbb{P}^{\hat{\widetilde{\sigma}}}(x_1 \geqslant \gamma_1 - N^{\xi}/2) \geqslant 1/2. \tag{7.11}$$

Fix $m \in \mathbb{N}$*. Then there is a small* $\chi > 0$ *such that for any* $\Lambda \subset [\![1, K^{\zeta}]\!]$*,* $|\Lambda| = m$*, and any smooth, compactly supported observable* $O : \mathbb{R}^m \to \mathbb{R}$*, we have*

$$\left|\mathbb{E}^{\sigma_{\mathbf{y}}} O\left(\left(j^{1/3}(x_j - \alpha_j)\right)_{j\in\Lambda}\right) - \mathbb{E}^{\widetilde{\sigma}_{\widetilde{\mathbf{y}}}} O\left(\left(j^{1/3}(x_j - \widetilde{\alpha}_j)\right)_{j\in\Lambda}\right)\right| \leqslant CN^{-\chi}. \tag{7.12}$$

The main tool for proving Theorem 7.1 is the interpolating measure between $\mu_{\mathbf{y}}$ and $\widetilde{\mu}_{\widetilde{\mathbf{y}}}$ which will be defined in Section 7.3.

7.2 Proof of Theorem 3.3 from Theorem 7.1

In order to prove Theorem 3.3, we will need a slight extension of Theorem 7.1 result we formulate now. We claim that Theorem 7.1 also holds if the measures are rescaled by an N-dependent factor, provided the rescaling factor is very close to one. More precisely, fix a small $\ell = \ell_N = O(K^{-1})$ and define the rescaled potential

$$V^*(x) = V\Big(\frac{x}{1+\ell}\Big)$$

and the cutoff potential $\sum_{i\in I}\Theta^*(x_i)$, where

$$\Theta^*(x) = \Theta\Big(N^{-\xi}x/(1+\ell)\Big).$$

From V^* and Θ^*, we define the rescaled measure σ^* by the formula (2.2) and (7.5). For any observable Q we clearly have the relation

$$\mathbb{E}^{\sigma^*}Q(\boldsymbol{\lambda}) = \mathbb{E}^{\sigma}Q((1+\ell)\boldsymbol{\lambda}).$$

Furthermore, the equilibrium density ϱ^* for the measure σ^* (defined by the variational principle (7.1); notice that it is independent of the cutoff Θ) satisfies

$$\varrho^*(x) = \frac{1}{1+\ell}\varrho\Big(\frac{x}{1+\ell}\Big).$$

Fix a boundary condition $\mathbf{y} = (y_{K+1},\dots,y_N) \in \mathcal{R}_K^{\#}\cap\mathcal{R}^*$ and define the rescaled boundary condition by $y_j^* = (1+\ell)y_j$ for all $j \geqslant K+1$. The conditional measure $\sigma^*_{\mathbf{y}^*}$ thus satisfies the relation

$$\mathbb{E}^{\sigma^*_{\mathbf{y}^*}}Q(x_1,\dots,x_K) = \mathbb{E}^{\sigma_{\mathbf{y}}}Q((1+\ell)x_1,\dots,(1+\ell)x_K) \tag{7.13}$$

and we also have $\alpha_j^* = (1+\ell)\alpha_j$. Now we will compare the measure $\widetilde{\sigma}_{\widetilde{\mathbf{y}}}$ with the rescaled conditional measure $\sigma^*_{\mathbf{y}^*}$ assuming that they have the same configurational interval $\widetilde{J} = J^*$, i.e., that $y^*_{K+1} = \widetilde{y}_{K+1}$ (in applications, we will choose ℓ in order to match these boundary conditions). Therefore, we would like to extend the validity of (7.12) to the rescaled measures, i.e., to conclude that

$$\left|\mathbb{E}^{\sigma^*_{\mathbf{y}^*}}O\left(\Big(j^{1/3}(x_j-\alpha_j^*)\Big)_{j\in\Lambda}\right) - \mathbb{E}^{\widetilde{\sigma}_{\widetilde{\mathbf{y}}}}O\left(\Big(j^{1/3}(x_j-\widetilde{\alpha}_j)\Big)_{j\in\Lambda}\right)\right| \leqslant N^{-\chi}. \tag{7.14}$$

Notice that if $\mathbf{y} \in \mathcal{R}_K^{\#}$ w.r.t. the measure σ then $\mathbf{y}^* \in \mathcal{R}_K^{\#}$ w.r.t. the measure σ^* by simple scaling. Again by scaling, we have

$$|\mathbb{E}^{\sigma^*_{\mathbf{y}^*}}x_j - \alpha_j^*| = (1+\ell)|\mathbb{E}^{\sigma_{\mathbf{y}}}x_j - \alpha_j| \leqslant CN^{\xi}j^{-1/3}$$

and thus (7.10) holds w.r.t. the measure $\sigma^*_{\mathbf{y}^*}$. Furthermore, we can check (7.11) holds with N^{ξ} replaced by $N^{\xi}(1+O(K^{-1}))$. Instead of (2.8), we now have

$$\varrho^*(t) = \frac{\varrho(t/(1+\ell))}{1+\ell} = \frac{\sqrt{t/(1+\ell)}[1+O(t/(1+\ell))]}{1+\ell} = \sqrt{t}\Big[1-\ell+O\big(t(1+O(\ell))\big)\Big]. \tag{7.15}$$

In order to prove (7.14), we need to check that the following proof of (7.12) holds with (2.8) replaced by (7.15) and the very minor change of (7.11) just mentioned. The task is straightforward and we will only remark on a small change in the proof near the equation (7.16).

We make another small observation. Similarly to the remark after Theorem 3.3, we can replace α_j by $\gamma_j = j^{2/3}(1+O\big[(j/N)^{2/3}\big])$ or simply by $j^{2/3}$ for the purpose of proving Theorem 7.1 as long as $j \leqslant K^{\zeta}$.

This follows from the smoothness of O, from (7.8) and from (7.15) that implies $\gamma_j = j^{2/3}(1+O\big[(j/N)^{2/3}\big)\big] = j^{2/3} + o(j^{-1/3})$. If we are dealing with the measure $\sigma^*_{\mathbf{y}^*}$, then for $j \in \Lambda$ there is $\chi > 0$ such that

$$\alpha^*_j = (1+\ell)\alpha_j = j^{2/3}(1+O\big[(j/N)^{2/3}\big)\big] + O\Big(N^{\xi}\,\frac{j^{2/3}}{K}\Big) + O(j^{2/3}K^{-1}) = j^{2/3} + O(j^{-1/3})N^{-\chi}. \tag{7.16}$$

Here we have used (7.8), $\ell = O(K^{-1})$ and ζ in the definition of the set Λ satisfying $\zeta < 1$.

Proof of Theorem 3.3. Under the condition (7.9), Theorem 3.3 would directly follow from Theorem 7.1. We now prove Theorem 3.3 in the general case. Suppose that $\mathbf{y}, \widetilde{\mathbf{y}} \in \mathcal{R}^{\#}_K \cap \mathcal{R}^*_K$ but the boundary condition (7.9) is not satisfied. Given these two boundary conditions, we define $\ell = \ell(\mathbf{y}, \widetilde{\mathbf{y}})$ by the formula

$$\widetilde{y}_{K+1} = (1+\ell)y_{K+1}. \tag{7.17}$$

Using this ℓ, we define the rescaled boundary conditions $y^*_j = (1+\ell)y_j$. Now we will compare the measure $\widetilde{\sigma}_{\widetilde{\mathbf{y}}}$ with the rescaled conditional measure $\sigma^*_{\mathbf{y}^*}$ which now has the same configurational interval $\widetilde{J} = J^*$. With the choice of ℓ in (7.17) and the rigidity estimate (2.11) , we can estimate ℓ by

$$|\ell| \leqslant N^{\xi}K^{-1},$$

where we have also used $\gamma_k \sim k^{2/3}$. From the rescaling identity (7.13) applied to an observable Q of special form, we have

$$\mathbb{E}^{\sigma^*_{\mathbf{y}^*}} O\bigg(\Big(j^{1/3}(x_j - \alpha^*_j)\Big)_{j\in\Lambda}\bigg) = \mathbb{E}^{\sigma_{\mathbf{y}}} O\bigg(\Big(j^{1/3}(1+\ell)\Big[x_j - \alpha_j\Big]\Big)_{j\in\Lambda}\bigg). \tag{7.18}$$

From the rigidity estimate (3.7) (notice we need to change to the microscopic coordinates), we have

$$\Big|j^{1/3}\ell(x_j - \alpha_j)\Big| \leqslant N^{2\xi}K^{-1} \leqslant N^{-\chi}. \tag{7.19}$$

since $2\xi + \chi < \delta$, see (3.11). We can use the smoothness of O to remove the $(1+\ell)$ factor on the right hand side of (7.18) by Taylor expansion at a negligible error. Using (7.14), we have proved that

$$\left|\mathbb{E}^{\sigma_{\mathbf{y}}} O\bigg(\Big(j^{1/3}(x_j - \alpha_j)\Big)_{j\in\Lambda}\bigg) - \mathbb{E}^{\widetilde{\sigma}_{\widetilde{\mathbf{y}}}} O\bigg(\Big(j^{1/3}(x_j - \widetilde{\alpha}_j)\Big)_{j\in\Lambda}\bigg)\right| \leqslant N^{-\chi} \tag{7.20}$$

and this proves Theorem 3.3. □

7.3 *Outline of the proof of Theorem 7.1*

The basic idea to prove (7.12) is to introduce a one-parameter family of interpolating measures between any two measures $\sigma_{\mathbf{y}}$ and $\widetilde{\sigma}_{\widetilde{\mathbf{y}}}$ with potentials $V_{\mathbf{y}}$ and $\widetilde{V}_{\widetilde{\mathbf{y}}}$ with fixed boundary conditions $\mathbf{y}$ and $\widetilde{\mathbf{y}}$ and possible two different external potentials V and $\widetilde{V}$. These measures are defined for any $0 \leqslant r \leqslant 1$ by

$$\omega = \omega^r_{\mathbf{y},\widetilde{\mathbf{y}}} \sim e^{-\beta N \mathcal{H}^r_{\mathbf{y},\widetilde{\mathbf{y}}}}, \qquad \mathcal{H}^r_{\mathbf{y},\widetilde{\mathbf{y}}}(\mathbf{x}) := \frac{2}{N}\sum_{i\in I}\Theta\big(N^{-\xi}x_i\big) + \sum_{i\in I}\frac{1}{2}V^r_{\mathbf{y},\widetilde{\mathbf{y}}}(x_i) - \frac{1}{N}\sum_{\substack{i,j\in I\\ i<j}}\log|x_j - x_i|, \tag{7.21}$$

with

$$V^r_{\mathbf{y},\widetilde{\mathbf{y}}}(x) := (1-r)V_{\mathbf{y}}(x) + r\widetilde{V}_{\widetilde{\mathbf{y}}}(x). \tag{7.22}$$

Notice that $\omega^{r=0}_{\mathbf{y},\widetilde{\mathbf{y}}} = \sigma_{\mathbf{y}}$ and $\omega^{r=1}_{\mathbf{y},\widetilde{\mathbf{y}}} = \widetilde{\sigma}_{\widetilde{\mathbf{y}}}$. Basic properties of the measure ω will be established in Section 8. Now we outline our main steps to prove (7.12).

Step 1. *Interpolation.* For any observable $Q(\mathbf{x})$, we rewrite the difference of the expectations of Q w.r.t. the two different local measures by

$$\mathbb{E}^{\sigma_{\mathbf{y}}} Q(\mathbf{x}) - \mathbb{E}^{\widetilde{\sigma}_{\widetilde{\mathbf{y}}}} Q(\mathbf{x}) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}r} \mathbb{E}^{\omega^r_{\mathbf{y},\widetilde{\mathbf{y}}}} Q(\mathbf{x}) \mathrm{d}r = \beta \int_0^1 \langle Q; h_0 \rangle_{\omega^r} \mathrm{d}r,$$

with

$$\begin{aligned} h_0(\mathbf{x}) : &= N \sum_{j \in I} [V_{\mathbf{y}}(x_j) - \widetilde{V}_{\widetilde{\mathbf{y}}}(x_j)] \\ &= \sum_{j \in I} \left[N\Big(V(x_j N^{-2/3}) - \widetilde{V}(x_j N^{-2/3}) \Big) - 2 \sum_{k \notin I} \Big(\log|x_j - y_k| - \log|x_j - \widetilde{y}_k| \Big) \right]. \end{aligned} \tag{7.23}$$

So the main goal is to show that for any $\omega = \omega^r_{\mathbf{y},\widetilde{\mathbf{y}}}$ with good boundary conditions we have

$$|\langle Q; h_0 \rangle_\omega| \leqslant N^{-\chi}. \tag{7.24}$$

This will hold for a certain class of observables Q that depend on a few coordinates near the left edge. The class of observables we are interested in have the form

$$Q(\mathbf{x}) := O\left(\Big(j^{1/3}(x_j - j^{2/3}) \Big)_{j \in \Lambda} \right). \tag{7.25}$$

Step 2. *Random walk representation.* For any smooth observables $F(\mathbf{x})$ and $Q(\mathbf{x})$ and any time $T > 0$ we have the following representation formula for the time dependent correlation function (see (9.3) for the precise statement):

$$\mathbb{E}^\omega Q(\mathbf{x})\, F(\mathbf{x}) - \mathbb{E}^\omega Q(\mathbf{x}(0)) F(\mathbf{x}(T)) = \frac{1}{2} \int_0^T \mathrm{d}S\, \mathbb{E}^\omega \sum_{b=1}^K \partial_b Q(\mathbf{x}(0)) \langle \nabla F(\mathbf{x}(S)), \mathbf{v}^b(S, \mathbf{x}(\cdot)) \rangle. \tag{7.26}$$

Here the path $\mathbf{x}(\cdot)$ is the solution of the reversible stochastic dynamics with equilibrium measure ω (see (8.1) later). We use the notation $\mathbb{E}^\omega$ also for the expectation with respect to the path measure starting from the initial distribution ω and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^K$. Furthermore, for any $b \in I$ and for any fixed path $\mathbf{x}(\cdot)$, the vector $\mathbf{v}^b(t) = \mathbf{v}^b(t, \mathbf{x}(\cdot)) \in \mathbb{R}^K$ is the solution to the equation

$$\partial_t \mathbf{v}^b(t) = -\mathcal{A}(t) \mathbf{v}^b(t), \quad t \geqslant 0, \qquad v^b_j(0) = \delta_{bj}.$$

The matrix $\mathcal{A}(t)$ depends on time through the path $\mathbf{x}(t)$, i.e., it is of the form $\mathcal{A}(t) = \widetilde{\mathcal{A}}(\mathbf{x}(t))$. It will be defined in (9.1) and it is related to the Hessian of the Hamiltonian $\mathcal{H}^r_{\mathbf{y},\widetilde{\mathbf{y}}}$ of the measure ω. Using rigidity estimates on the path $\mathbf{x}(\cdot)$, we will show that with very high probability the matrix elements of $\mathcal{A}(t)$ satisfy the time-independent lower bound

$$\mathcal{A}(t)_{ij} \geqslant \frac{1}{(i^{2/3} - j^{2/3})^2} + \delta_{ij} \frac{K^{2/3}}{K^{2/3} - j^{2/3}}, \tag{7.27}$$

up to irrelevant factors (see (10.14), (10.15)).

We apply the random walk representation (7.26) for $T \sim K^{1/3}$ and $F = h_0$. This is sufficient since the time to equilibrium for the $\mathbf{x}(t)$ process is of order $K^{1/3}$, which will be guaranteed by convexity properties of the Hamiltonian of the measure ω (Lemma 8.1).

Step 3. If the coefficient matrix $\mathcal{A}(t)$ satisfies (7.27), then the semigroup associated with the equation

$$\partial_t \mathbf{u}(t) = -\mathcal{A}(t)\mathbf{u}(t) \tag{7.28}$$

has good $L^p \to L^q$ decay estimates (Proposition 10.4) that follow from energy method and a new Sobolev inequality (Proposition 10.5). Rigidity estimates w.r.t. ω (Lemma 8.2) will ensure that the bound (7.27) holds with very high probability. The $L^p \to L^q$ decay estimates together with the bound

$$|\partial_j h_0(\mathbf{x})| \lesssim \frac{K^{1/3}}{K+1-j}, \qquad j \in I,$$

that also follows from rigidity, will allow us to reduce the upper limit in the time integration in (7.26) from $T \sim K^{1/3}$ to $\widetilde{T} \sim K^{1/6}$ in (7.26). The necessary rigidity estimate w.r.t. ω is obtained by interpolating between the rigidity estimates for $\sigma_{\mathbf{y}}$ and $\widetilde{\sigma}_{\widetilde{\mathbf{y}}}$.

Step 4. Finally, we also have a time dependent version of the $L^\alpha \to L^\infty, \alpha > 1$, decay estimate that follows from a different Sobolev inequality (see Theorem 10.8). More precisely, in Lemma 10.7 we will show that if the matrix elements $\mathcal{A}_{ij}(t)$ satisfy (7.27), then for the M-th coordinate of the solution to (7.28) we have for any $\alpha > 1$

$$\int_0^t |u_M(s)|^\alpha \mathrm{d}s \leqslant C_\alpha M^{-2/3}(t+1)\|\mathbf{u}(0)\|_\alpha^\alpha, \qquad M \in I, \quad t > 0,$$

(up to irrelevant factors). We will apply this bound with $\alpha = 1 + \varepsilon$ to control the remaining time integration from 0 to $\widetilde{T}$ in (7.26).

8 Properties of the interpolating measure

In this section we establish the necessary apriori results for ω, defined in (7.21). We start with its speed to equilibrium from a convexity bound on the Hessian. The measure ω defines a Dirichlet form D^ω and its generator $\mathcal{L}^\omega$ in the usual way:

$$-\langle f, \mathcal{L}^\omega f\rangle_\omega = -\int f \mathcal{L}^\omega f \mathrm{d}\omega = D^\omega(f) = \frac{1}{2}\int |\nabla f|^2 \mathrm{d}\omega,$$

where

$$\mathcal{L}^\omega = \frac{1}{2}\sum_{i \in I}\Big[\partial_i^2 + \beta\Big\{ -4N^{-\xi}\Theta'\big(N^{-\xi}x_i\big) - N(V^r_{\mathbf{y},\widetilde{\mathbf{y}}})'(x_i) + \sum_{j \neq i}\frac{1}{x_i - x_j}\Big\}\partial_i\Big].$$

Note that in the context of studying the dynamics near the edge in the microscopic coordinates, the natural Dirichlet form is defined without the $1/N$ prefactor in contrast to (5.5) and (5.2), where the scaling was dictated by the bulk.

Finally, let $\mathbf{x}(t)$ denote the corresponding stochastic process (local Dyson Brownian motion), given by

$$\mathrm{d}x_i = \mathrm{d}B_i + \beta\Big[-2N^{1-\xi}\Theta'\big(N^{-\xi}x_i\big) - \frac{N}{2}(V^r_{\mathbf{y},\widetilde{\mathbf{y}}})'(x_i) + \frac{1}{2}\sum_{j \neq i}\frac{1}{x_i - x_j}\Big]\mathrm{d}t, \qquad i \in I, \tag{8.1}$$

where $(B_1, \dots, B_K)$ is a family of independent standard Brownian motions. With a slight abuse of notations, when we talk about the process, we will use $\mathbb{P}^\omega$ and $\mathbb{E}^\omega$ to denote the probability and expectation w.r.t. this dynamics with initial data ω, i.e., in equilibrium. This dynamical point of view gives rise to a representation for the correlation functions in terms random walks in random environment. Note that $\beta \geqslant 1$ is needed for the well-posedness of (8.1). From the Hessian bound (7.7) and the Bakry-Émery criterion we have proved the following result:

Lemma 8.1. *Let ξ be any fixed positive constant and assume K satisfies* (3.1)*. Let $\mathbf{y}, \widetilde{\mathbf{y}} \in \mathcal{R} = \mathcal{R}_K(\xi)$, $r \in [0,1]$ and set $\omega = \omega^r_{\mathbf{y},\widetilde{\mathbf{y}}}$. Then the measure $\omega = \omega^r_{\mathbf{y},\widetilde{\mathbf{y}}}$ satisfies the logarithmic Sobolev inequality*

$$S(g\omega|\omega) \leqslant CK^{1/3} D^{\omega}(\sqrt{g})$$

and the time to equilibrium for the dynamics $\mathcal{L}^{\omega}$ is at most of order $K^{1/3}$. □

Next, we formulate the rigidity and level repulsion bounds for ω.

Lemma 8.2 (Rigidity and level repulsion for ω)**.** *Let ξ be any fixed positive constant and assume K satisfies* (3.1)*. Let $\mathbf{y}, \widetilde{\mathbf{y}} \in \mathcal{R} = \mathcal{R}_K(\xi)$, $r \in [0,1]$ and set $\omega = \omega^r_{\mathbf{y},\widetilde{\mathbf{y}}}$. Recall also the definition of α_i from* (7.4)*. Then the following holds:*

(i) [Rigidity] There is a constant $c > 0$ such that

$$\mathbb{P}^{\omega}\big(|x_i - \alpha_i| \geqslant N^{C_2\xi} i^{-1/3} u\big) \leqslant e^{-cu^2}, \qquad i \in I, \quad u > 0. \tag{8.2}$$

(ii) [Level repulsion] For any $s > 0$ we have

$$\mathbb{P}^{\omega}[y_{K+1} - x_K \leqslant sK^{-1/3}] \leqslant C\left(K^2 s\right)^{\beta+1}, \tag{8.3}$$

$$\mathbb{P}^{\omega}[y_{K+1} - x_K \leqslant sK^{-1/3}] \leqslant C\left(N^{C\xi} s\right)^{\beta+1} + e^{-N^c}. \tag{8.4}$$

(iii) We also have

$$\mathbb{E}^{\omega}|\log(y_{K+1} - x_K)| \leqslant CN^{C\xi}, \tag{8.5}$$

$$\mathbb{E}^{\omega}\frac{1}{|y_{K+1} - x_K|^q} \leqslant C_q N^{C\xi} K^{q/3}, \qquad q < \beta + 1. \tag{8.6}$$

The key to translate the rigidity estimate of the measures $\sigma_{\mathbf{y}}$ and $\sigma_{\widetilde{\mathbf{y}}}$ to the measure $\omega = \omega^r_{\mathbf{y},\widetilde{\mathbf{y}}}$ is the following lemma.

Lemma 8.3. *Let K satisfy* (3.1) *and $\mathbf{y}, \widetilde{\mathbf{y}} \in \mathcal{R}_K(\xi)$. Consider the local equilibrium measure $\sigma_{\mathbf{y}}$ defined in* (7.6) *and assume that* (7.10) *is satisfied. Let $\omega^r_{\mathbf{y},\widetilde{\mathbf{y}}}$ be the measure defined in* (7.21)*. Recall that α_k denote the equidistant points in J, see* (7.4)*. Then there exists a constant C, independent of ξ, such that*

$$\mathbb{E}^{\omega^r_{\mathbf{y},\widetilde{\mathbf{y}}}} |x_j - \alpha_j| \leqslant CN^{C\xi}. \tag{8.7}$$

Proof of Lemma 8.3. We first recall the following estimate on the entropy from Lemma 6.9 of [25].

Lemma 8.4. *Suppose μ_1 is a probability measure and $\omega = Z^{-1} e^g \mathrm{d}\mu_1$ for some function $g \in L^1(\mathrm{d}\mu_1)$ with $e^g \in L^1(\mathrm{d}\mu_1)$ and normalization Z. Then we can bound the entropy by*

$$S := S(\omega|\mu_1) = \mathbb{E}^{\omega} g - \log \mathbb{E}^{\mu_1} e^g \leqslant \mathbb{E}^{\omega} g - \mathbb{E}^{\mu_1} g.$$

Consider two probability measures $\mathrm{d}\mu_i = Z_i^{-1} e^{-H_i} \mathrm{d}\mathbf{x}$, $i = 1, 2$. Denote by g the function

$$g = r(H_1 - H_2), \quad 0 < r < 1,$$

and set $\omega = Z^{-1}e^g d\mu_1$ as above. Then we can bound the entropy by

$$\min(S(\omega|\mu_1), S(\omega|\mu_2)) \leqslant \Big[\mathbb{E}^{\mu_2} - \mathbb{E}^{\mu_1}\Big](H_1 - H_2).$$

We now apply this lemma with $\mu_2 = \widetilde{\sigma}_{\widetilde{\mathbf{y}}}$ and $\mu_1 = \sigma_{\mathbf{y}}$ to prove that

$$\min[S(\omega^r_{\mathbf{y},\widetilde{\mathbf{y}}}|\sigma_{\mathbf{y}}), S(\omega^r_{\mathbf{y},\widetilde{\mathbf{y}}}|\widetilde{\sigma}_{\widetilde{\mathbf{y}}})] \leqslant N^{C\xi}. \tag{8.8}$$

To see this, by definition of g and the rigidity estimate (2.11), we have

$$\begin{aligned}
\mathbb{E}^{\mu_2} g - \mathbb{E}^{\mu_1} g &= \frac{r}{2}\Big[\mathbb{E}^{\mu_2} - \mathbb{E}^{\mu_1}\Big]\sum_{i\in I}\Big[V_{\mathbf{y}}(x_i) - \widetilde{V}_{\widetilde{\mathbf{y}}}(x_i)\Big] \\
&= \frac{r}{2}\Big[\mathbb{E}^{\mu_2} - \mathbb{E}^{\mu_1}\Big]\sum_{i\in I}\Big[V_{\mathbf{y}}(x_i) - \widetilde{V}_{\widetilde{\mathbf{y}}}(x_i) - \big(V_{\mathbf{y}}(\alpha_i) - \widetilde{V}_{\widetilde{\mathbf{y}}}(\alpha_i)\big)\Big] \\
&= \frac{r}{2}\Big[\mathbb{E}^{\mu_2} - \mathbb{E}^{\mu_1}\Big]\sum_{i\in I}\int_0^1 \mathrm{d}s\Big[V'_{\mathbf{y}}(s\alpha_i + (1-s)x_i) - \widetilde{V}'_{\widetilde{\mathbf{y}}}(s\alpha_i + (1-s)x_i)\Big](x_i - \alpha_i) \\
&= \Big[\mathbb{E}^{\mu_2} + \mathbb{E}^{\mu_1}\Big]O\Big(\sum_{i\in I}\sup_{s\in[0,1]}\frac{K^\xi}{|s\alpha_i + (1-s)x_i - y_{K+1}|}|x_i - \alpha_i|\Big) \leqslant N^{C\xi}.
\end{aligned} \tag{8.9}$$

In the last step we used the rigidity (3.7) to see that with a very high μ_1- or μ_2-probability the numbers $s\alpha_i + (1-s)x_i \sim \alpha_i$ are equidistant up to an additive error K^ξ if i is away from the boundary, i.e., $i \leqslant K - K^{C\xi}$, see (3.7). For indices near the boundary, $i \geqslant K - K^{C\xi}$, we used $|s\alpha_i + (1-s)x_i| \geqslant c\min\{1, |x_K - y_{K+1}|\}$ and the rigidity $|x_i - \alpha_i| \leqslant N^{C\xi}K^{-1/3}$. The bound (8.6) guarantees that the short distance singularity $|x_K - y_{K+1}|^{-1}$ has an $\mathbb{E}^{\mu_{1,2}}$ expectation that is bounded by $CN^{C\xi}K^{1/3}$, which gives (8.9).

We now assume that (8.8) holds with the choice of $S(\omega^r_{\mathbf{y},\widetilde{\mathbf{y}}}|\sigma_{\mathbf{y}})$ for simplicity of notation. By the entropy inequality, we have

$$\mathbb{E}^{\omega^r_{\mathbf{y},\widetilde{\mathbf{y}}}}|x_i - \gamma_i| \leqslant N^{\xi+\varepsilon}\log\mathbb{E}^{\sigma_{\mathbf{y}}}e^{N^{-\xi-\varepsilon}|x_i-\gamma_i|} + N^{C\xi}N^{\xi+\varepsilon} \leqslant N^{C\xi}.$$

This proves Lemma 8.3. □

Proof of Lemma 8.2. Given (8.7), the proof of (8.2) follows from the argument in the proof of Theorem 3.1. Once the rigidity bound (8.2) is proved, we can follow the proof of Theorem 3.2 to obtain the repulsion estimates (8.3)-(8.4). The only modification is that we use the potential $V^r_{\mathbf{y},\widetilde{\mathbf{y}}}$ of the measure $\omega = \omega^r_{\mathbf{y},\widetilde{\mathbf{y}}}$ (see (7.22)) instead of $V_{\mathbf{y}}$. The analogue of $V^*_{\mathbf{y}}$ (see (D.5)) can be directly defined for $V^r_{\mathbf{y},\widetilde{\mathbf{y}}}$ as

$$[V^r_{\mathbf{y},\widetilde{\mathbf{y}}}]^*(x) = (1-r)V^*_{\mathbf{y}}(x) + r\widetilde{V}^*_{\widetilde{\mathbf{y}}}(x). \tag{8.10}$$

Formula (D.4) will be slightly modified, e.g. the factor $(y_- + (1-\varphi)(w_j - y_-) - y_k)^\beta$ will be replaced with $(y_- + (1-\varphi)(w_j - y_-) - y_k)^{(1-r)\beta}(y_- + (1-\varphi)(w_j - y_-) - \widetilde{y}_k)^{r\beta}$, but it does not change the estimates. Similarly, the necessary bound (C.3) for the potential $[V^r_{\mathbf{y},\widetilde{\mathbf{y}}}]^*$ easily follows from (8.10) and the same bounds on $V^*_{\mathbf{y}}$ and $V^*_{\widetilde{\mathbf{y}}}$. Finally, (8.5) and (8.6) are trivial consequences of (8.3) and (8.4). □

9 Random walk representation for the correlation function

The first step to prove (7.24) is to use the random walk representation formula from Proposition 7.1 of [25] which we restate in Proposition 9.1 below. This formula in a lattice setting was given in Proposition 2.2 of [17] (see also Proposition 3.1 in [33]). The random walk representation already appeared in the earlier paper of Naddaf and Spencer [42], which was a probabilistic formulation of the idea of Helffer and Sjöstrand [35].

Fix $S > 0$ and $\mathbf{x} \in J^K = J^I_{\mathbf{y}}$. Let $\mathbf{x}(s)$ be the solution to (8.1) with initial condition $\mathbf{x}(0) = \mathbf{x}$. Let $\mathbb{E}^{\mathbf{x}}$ denote the expectation with respect to this path measure. With a slight abuse of notations, we will use $\mathbb{P}^\omega$ and $\mathbb{E}^\omega$ to denote the probability and expectation with respect to the path measure of the solution to (8.1) with initial condition $\mathbf{x}$ distributed by ω.

For any fixed path $\mathbf{x}(\cdot) := \{\mathbf{x}(s) \ : \ s \in [0, S]\}$ we define the following operator ($K \times K$ matrix) acting on K-vectors $\mathbf{u} \in \mathbb{R}^K$ indexed by the set I;

$$\mathcal{A}(s) = \widetilde{\mathcal{A}}(\mathbf{x}(s)), \qquad \widetilde{\mathcal{A}} = \widetilde{\mathcal{B}} + \widetilde{\mathcal{W}}, \tag{9.1}$$

with actions

$$[\widetilde{\mathcal{B}}(\mathbf{x})\mathbf{u}]_i := \frac{1}{2}\sum_{j\in I} \frac{1}{(x_i - x_j)^2}(u_i - u_j), \qquad [\widetilde{\mathcal{W}}(\mathbf{x})\mathbf{u}]_i = \mathcal{W}_i u_i \qquad i \in I,$$

where we defined

$$\widetilde{\mathcal{W}}_i(\mathbf{x}) = 2N^{1-2\xi}\Theta''(N^{-\xi}x_i) + \frac{N^{-1/3}}{2}\big[(1-r)V''(x_i N^{-2/3}) + r\widetilde{V}''(x_i N^{-2/3})\big] + \frac{1}{2}\sum_{k\notin I}\Big[\frac{1-r}{(y_k - x_i)^2} + \frac{r}{(\widetilde{y}_k - x_i)^2}\Big]. \tag{9.2}$$

(Notice that $W_i(\mathbf{x})$ depends only on x_i).

Proposition 9.1. *For any smooth functions $F : J^K \to \mathbb{R}$ and $Q : J^K \to \mathbb{R}$ and any time $T > 0$ we have*

$$\mathbb{E}^\omega Q(\mathbf{x})\, F(\mathbf{x}) - \mathbb{E}^\omega Q(\mathbf{x}(0))F(\mathbf{x}(T)) = \frac{1}{2}\int_0^T \mathrm{d}S \int \omega(\mathrm{d}\mathbf{x}) \sum_{a,b=1}^K \partial_b Q(\mathbf{x})\mathbb{E}^{\mathbf{x}}\partial_a F(\mathbf{x}(S))v_a^b(S, \mathbf{x}(\cdot)). \tag{9.3}$$

Here for any $S > 0$ and for any path $\{\mathbf{x}(s) \in J^K \ : \ s \in [0, S]\}$, we define $\mathbf{v}^b(t) = \mathbf{v}^b(t, \mathbf{x}(\cdot))$ as the solution to the equation

$$\partial_t \mathbf{v}^b(t) = -\mathcal{A}(t)\mathbf{v}^b(t), \quad t \in [0, S], \qquad v_a^b(0) = \delta_{ba}. \tag{9.4}$$

The dependence of $\mathbf{v}^b$ on the path $\mathbf{x}(\cdot)$ is present via the dependence $\mathcal{A}(t) = \widetilde{\mathcal{A}}(\mathbf{x}(t))$. In other words, $v_a^b(t)$ is the fundamental solution of the heat semigroup $\partial_s + \mathcal{A}(s)$.

10 Proof of Theorem 7.1

From now on we assume the conditions of Theorem 7.1. In particular we are given some $\xi > 0$ and we assume that the boundary conditions satisfy $\mathbf{y}, \widetilde{\mathbf{y}} \in \mathcal{R}^{\#}(\xi)$ and (7.10).

10.1 *First time cutoff*

We now start to estimate the correlation function in (7.24). We first apply the formula (9.3) with F replaced by h_0 defined in (7.23) so that

$$\mathbb{E}^\omega Q(\mathbf{x})h_0(\mathbf{x}) - \mathbb{E}^\omega Q(\mathbf{x}(0))h_0(\mathbf{x}(T)) = \frac{1}{2}\int_0^T \mathrm{d}S \int \omega(\mathrm{d}\mathbf{x}) \sum_{a,b=1}^K \partial_b Q(\mathbf{x})\mathbb{E}^{\mathbf{x}}\partial_a h_0(\mathbf{x}(S))v_a^b(S,\mathbf{x}(\cdot)).$$

We collect information on h_0 in the following lemma:

Lemma 10.1. *Let K satisfy*

$$N^\delta \leqslant K \leqslant N^{2/5-\delta} \tag{10.1}$$

for some small $\delta > 0$ and let $\mathbf{y}, \widetilde{\mathbf{y}} \in \mathcal{R}^\#(\xi)$. Then for any $\kappa < \beta + 1$ we have

$$\mathbb{E}^\omega |h_0(\mathbf{x})|^\kappa \leqslant C_\kappa K^2 N. \tag{10.2}$$

Furthermore, if $\mathbf{x}$ satisfies

$$\max_{j\in I} j^{1/3}|x_j - \alpha_j| \leqslant N^{C_3\xi}, \tag{10.3}$$

then

$$|\partial_j h_0(\mathbf{x})| \leqslant \frac{CN^{C\xi}K^{1/3}}{K+1-j}, \qquad j \in I. \tag{10.4}$$

In particular, we have the L^1-bound

$$\sum_j |\partial_j h_0(\mathbf{x})| \leqslant CN^{C\xi}K^{1/3}. \tag{10.5}$$

Proof. The bound (10.2) follows from (8.3), while (10.4) will be proven in Appendix C. □

Since the time to equilibrium of the $\mathcal{L}^\omega$ dynamics is of order $K^{1/3}$ (see Lemma 8.1), by choosing

$$T := CK^{1/3}\log N$$

with a large constant C, we have

$$\left|\mathbb{E}^\omega Q(\mathbf{x}(0))h_0(\mathbf{x}(T)) - \mathbb{E}^\omega Q(\mathbf{x})\, \mathbb{E}^\omega h_0(\mathbf{x})\right| \leqslant N^{-C}. \tag{10.6}$$

In proving this relation, we use a cutoff argument. Although h_0 is singular and it is not in $L^2(\omega)$, we can write $h_0 = h_< + h_>$, $h_<(x) := h_0(x)\mathbb{1}(h_0(x) \leqslant N^C)$. By (10.2) the probability $\mathbb{P}^\omega(h_0 \geqslant N^C)$ and hence the contribution of $h_>$ to (10.6) are negligible. The function $h_<$ is in $L^2(\mathrm{d}\omega)$, so we can use the spectral gap of order $CK^{1/3}$ (Lemma 8.1) to show that the contribution of the $h_<$ part to (10.6) is also negligible. We can thus represent the correlation function as

$$\langle Q; h_0\rangle_\omega = \frac{1}{2}\int_0^T \mathrm{d}S \int \omega(\mathrm{d}\mathbf{x}) \sum_{a,b=1}^K \partial_b Q(\mathbf{x})\mathbb{E}^{\mathbf{x}}\partial_a h_0(\mathbf{x}(S))v_a^b(S,\mathbf{x}(\cdot)) + O(N^{-C}). \tag{10.7}$$

10.2 Set of good paths

We have a good control on the solution to (9.4) if the coordinates of the trajectory $\mathbf{x}(\cdot)$ remain close to the classical locations $(\alpha_1, \dots, \alpha_K)$. Setting a constant $C_3 > C_2$ (C_2 is the constant in (8.2)), for any T we thus define the set of "good" path as:

$$\mathcal{G}_T := \Big\{ \sup_{0 \leqslant s \leqslant T} \max_{j \in I} j^{1/3} |x_j(s) - \alpha_j| \leqslant N^{C_3 \xi} \Big\}, \tag{10.8}$$

where α_j is given by (7.4).

Lemma 10.2. *Assume that the rigidity estimate (8.2) holds for the measure ω. For the cutoff time $T = CK^{1/3} \log N$, there exists a positive constant θ, depending on ξ, such that*

$$\mathbb{P}^\omega(\mathcal{G}_T^c) \leqslant e^{-N^\theta}. \tag{10.9}$$

Proof. We first recall the following result of Kipnis-Varadhan [37]:

Lemma 10.3. *For any process with a reversible measure ω and Dirichlet form $D^\omega(f) = \frac{1}{2}\int |\nabla f|^2 \mathrm{d}\omega$, we have*

$$\mathbb{P}^\omega(\sup_{0 \leqslant s \leqslant T} |f(\mathbf{x}(s))| \geqslant \ell) \leqslant \frac{1}{\ell} \sqrt{\|f\|_2^2 + TD^\omega(f)}. \tag{10.10}$$

To apply this lemma, let $f(\mathbf{x}) = g(x_j)$ with

$$g(x) = e^{N^{-C\xi}(x_j - \alpha_j) j^{1/3}}.$$

From the rigidity estimate (8.2)

$$\|f\|_2^2 + TD^\omega(f) \leqslant \Big[1 + T\big(N^{-C\xi} j^{1/3}\big)^2\Big] \|f\|_2^2 \leqslant CK(\log N) \int_{\mathbb{R}} e^{|u|} e^{-cu^2} \mathrm{d}u \leqslant CK^2.$$

From (10.10), we have for any $c > 0$

$$\mathbb{P}(\sup_{0 \leqslant s \leqslant T} N^{-C\xi}(x_j(s) - \alpha_j) j^{1/3} \geqslant N^c) \leqslant \mathbb{P}(\sup_{0 \leqslant s \leqslant T} |g(x_j(s))| \geqslant e^{N^c}) \leqslant CK^2 e^{-N^c}.$$

Similarly, we can prove

$$\mathbb{P}(\sup_{0 \leqslant s \leqslant T} N^{-C\xi}(\alpha_j - x_j) j^{1/3} \geqslant N^c) \leqslant Ce^{-N^c}.$$

This proves Lemma 10.2. □

10.3 Restriction to the set $\mathcal{G}_T$

Now we show that the expectation (10.7) can be restricted to the good set $\mathcal{G} := \mathcal{G}_T$ with a small error. With a slight abuse of notations we use $\mathcal{G}$ also to denote the characteristic function of the set $\mathcal{G}$. For a fixed S and for a fixed $b \in I$ we can estimate the contribution of the $\mathcal{G}^c$ by

$$\frac{1}{2} \int \big|\partial_b Q(\mathbf{x})\big| \sum_{a=1}^{K} \mathbb{E}^{\mathbf{x}} \Big[\mathcal{G}^c |\partial_a h_0(\mathbf{x}(S))| v_a^b(S, \mathbf{x}(\cdot)) \Big] \omega(\mathrm{d}\mathbf{x}).$$

Since $\mathcal{A} \geqslant 0$ as a $K \times K$ matrix, the equation (9.4) is contraction in L^2. Clearly $\mathcal{A}$ is a contraction in L^1 as well, hence it is a contraction in any L^q, $1 \leqslant q \leqslant 2$, by interpolation. By the Hölder inequality and the L^q-contraction for some $1 < q < 2$, we have $\sum_a |v_a^b(S, \mathbf{x}(\cdot))|^q \leqslant \sum_a |v_a^b(0, \mathbf{x}(\cdot))|^q = 1$, so we get

$$\begin{aligned}
\mathbb{E}^\omega \mathcal{G}^c \Big| \sum_{a=1}^K |\partial_a h_0(\mathbf{x}(S))| v_a^b(S, \mathbf{x}(\cdot)) \Big| &\leqslant \big[\mathbb{E}^\omega \mathcal{G}^c\big]^{q/(q-1)} \Big[\mathbb{E}^\omega \Big| \sum_{a=1}^K |\partial_a h_0(\mathbf{x}(S))| v_a^b(S, \mathbf{x}(\cdot)) \Big|^q \Big]^{1/q} \\
&\leqslant \big[\mathbb{P}^\omega \mathcal{G}^c\big]^{q/(q-1)} \Big[\mathbb{E}^\omega \sum_{a=1}^K |\partial_a h_0(\mathbf{x}(S))|^q \Big]^{1/q} \\
&\leqslant C e^{-cN^\theta} K \cdot \max_a \Big[\mathbb{E}^\omega |\partial_a h_0(\mathbf{x})|^q \Big]^{1/q} \leqslant e^{-cN^{\theta_4}}
\end{aligned}$$

with some $\theta_4 > 0$. Here we used (10.9) for the first factor. In the second factor, after the invariance of the dynamics, we used the explicit form of h_0 (7.23) and the level repulsion bound (8.6):

$$\mathbb{E}^\omega |\partial_a h_0|^q \leqslant CN^{q/3} + C \max_a \mathbb{E}^\omega \Big[\sum_{k \notin I} \frac{1}{|x_a - y_k|^q} \Big] \leqslant CN.$$

Therefore, from (10.7) we conclude that

$$\langle Q; h_0 \rangle_\omega = \frac{1}{2} \int_0^T \mathrm{d}S \int \omega(\mathrm{d}\mathbf{x}) \sum_{a,b=1}^K \partial_b Q(\mathbf{x}) \mathbb{E}^{\mathbf{x}} \Big[\mathcal{G}\ \partial_a h_0(\mathbf{x}(S))\ v_a^b(S, \mathbf{x}(\cdot)) \Big] + O(N^{-C}). \tag{10.11}$$

In the next step we will reduce the upper limit of the time integration from $T \sim K^{1/3}$ to $\widetilde{T} \sim K^{1/6}$. This reduction uses effective $L^p \to L^q$ bounds on the solution to (9.4) that we will obtain with energy method and Nash-type argument.

10.4 Energy method and the evolution equation on the good set $\mathcal{G}$

In order to study the evolution equation (9.4) with $\mathbf{x}(\cdot)$ in the good set $\mathcal{G}$, we consider the following general evolution equation

$$\partial_s \mathbf{u}(s) = -\mathcal{A}(s)\mathbf{u}(s), \qquad \mathbf{u}(s) \in \mathbb{R}^I = \mathbb{R}^K, \qquad \mathbf{u}(0) = \mathbf{u}_0. \tag{10.12}$$

Here $\mathcal{A}$ and $\mathcal{B}$ are time dependent matrices of the form

$$\mathcal{A}(s) = \mathcal{B}(s) + \mathcal{W}(s), \qquad \text{with} \quad [\mathcal{B}(s)\mathbf{u}]_i = \sum_j B_{ij}(s)(u_i - u_j), \qquad [\mathcal{W}(s)\mathbf{u}]_i = W_i(s)u_i. \tag{10.13}$$

For x_i, x_j satisfying the rigidity bound defined in the good path (10.8) we have

$$\widetilde{\mathcal{B}}(\mathbf{x})_{ij} := \frac{1}{(x_i - x_j)^2} \geqslant \frac{N^{-C\xi}}{(i^{2/3} - j^{2/3})^2} \tag{10.14}$$

for some constant C. Similarly, for $\mathbf{y} \in \mathcal{R}$ and x_i satisfying the rigidity bound defined in the good path (10.8) we have

$$\widetilde{\mathcal{W}}_i(\mathbf{x}) \geqslant \sum_{\ell > K} \frac{1}{(x_i - y_\ell)^2} \geqslant \frac{K^{1/3} N^{-C\xi}}{d_i}, \qquad d_j := (K+1)^{2/3} - j^{2/3}, \qquad j \in I, \tag{10.15}$$

where we have used the definition of $\widetilde{W}$ in (9.2) and $\Theta'' \geqslant 0$.

Denote the L^p-norm of a vector $\mathbf{u} = \{u_j \; : \; j \in I\}$ by

$$\|\mathbf{u}\|_p = \Big(\sum_{j \in I} |u_j|^p \Big)^{1/p}.$$

We have the following decay estimate.

Proposition 10.4. *Let $\mathcal{A}$ be given in* (10.13) *and consider the evolution equation* (10.12). *Fix $S > 0$. Suppose that for some constant b the coefficients of $\mathcal{A}$ satisfy*

$$B_{jk}(s) \geqslant \frac{b}{(j^{2/3} - k^{2/3})^2}, \quad 0 \leqslant s \leqslant S, \quad j \neq k \in I, \tag{10.16}$$

and

$$W_j(s) \geqslant \frac{bK^{1/3}}{d_j}, \qquad d_j := (K+1)^{2/3} - j^{2/3}, \quad j \in I, \quad 0 \leqslant s \leqslant S. \tag{10.17}$$

Then for any $1 \leqslant p \leqslant q \leqslant \infty$ and for any small $\eta > 0$ we have the decay estimate

$$\|\mathbf{u}(s)\|_q \leqslant C(p,q,\eta) \Big[(K^{-\frac{2}{3}\eta} sb)^{-(\frac{3}{p} - \frac{3}{q})} \Big]^{1-6\eta} \|\mathbf{u}(0)\|_p, \qquad 0 < s \leqslant S. \tag{10.18}$$

Proof. We consider only the case $b = 1$, the general case follows from scaling. We follow the idea of Nash and start from the L^2-identity

$$\partial_s \|\mathbf{u}(s)\|_2^2 = -2\mathfrak{a}(s)[\mathbf{u}(s), \mathbf{u}(s)],$$

where $\mathfrak{a}(s)[\mathbf{u}, \mathbf{v}] := \sum_i u_i [\mathcal{A}(s)\mathbf{v}]_i$ is the quadratic form of $\mathcal{A}(s)$. For each s we can extend $\mathbf{u}(s) : I \to \mathbb{R}^K$ to a function $\widetilde{\mathbf{u}}(s)$: on $\mathbb{Z}_+$ by defining $\widetilde{u}_j(s) = u_j(s)$ for $j \leqslant K$ and $\widetilde{u}_j(s) = 0$ for $j > K$. Dropping the time argument, we have, by the estimates (10.16) and (10.17) with $b = 1$,

$$2\mathfrak{a}[\mathbf{u}, \mathbf{u}] \geqslant c \sum_{i,j \in \mathbb{Z}_+} \frac{(\widetilde{u}_i - \widetilde{u}_j)^2}{(i^{2/3} - j^{2/3})^2} \geqslant K^{-\frac{2}{3}\eta} \sum_{i,j \in \mathbb{Z}_+} \frac{(\widetilde{u}_i - \widetilde{u}_j)^2}{|i^{2/3} - j^{2/3}|^{2-\eta}} \geqslant c_\eta K^{-\frac{2}{3}\eta} \|\widetilde{\mathbf{u}}\|_p^2 = c_\eta K^{-\frac{2}{3}\eta} \|\mathbf{u}\|_p^2, \quad p := \frac{3}{1+\eta}, \tag{10.19}$$

with some positive constant c_η. In the first inequality, to estimate the W term, we have used that

$$\sum_{i > K} \frac{1}{(i^{2/3} - j^{2/3})^2} \leqslant \frac{CK^{1/3}}{d_j} \leqslant CW_j, \quad j \leqslant K,$$

to estimate the summation in (10.19) when one of the indices i, j is bigger than K. In the second inequality we used that

$$|i^{2/3} - j^{2/3}|^\eta \leqslant K^{\frac{2}{3}\eta}$$

for any $i, j \leqslant K$ which is the support of $\widetilde{u}$. In the third inequality we used the discrete version of the following Sobolev type inequality that will be proved in Appendix B.

Proposition 10.5. *We will formulate our result both in the continuous and in the discrete setting.*

(i) *Continuous version. For any small $\eta > 0$ there exists $c_\eta > 0$ such that for any real function f defined on $\mathbb{R}_+$, we have*

$$\int_0^\infty \int_0^\infty \frac{(f(x) - f(y))^2}{|x^{2/3} - y^{2/3}|^{2-\eta}} \mathrm{d}x\mathrm{d}y \geqslant c_\eta \Big(\int_0^\infty |f(x)|^p \mathrm{d}x \Big)^{2/p}, \qquad p := \frac{3}{1+\eta}. \tag{10.20}$$

(ii) Discrete version. For any small $\eta>0$ there exists $c_\eta>0$ such that for any sequence $\mathbf{u}=(u_1,u_2,\dots)$ we have

$$\sum_{i\neq j\in\mathbb{Z}_+}\frac{(u_i-u_j)^2}{|i^{2/3}-j^{2/3}|^{2-\eta}}\geqslant c_\eta\Big(\sum_{i\in\mathbb{Z}_+}|u_i|^p\Big)^{2/p}=c_\eta\|\mathbf{u}\|_p^2. \tag{10.21}$$

We now return to the proof of Proposition 10.4. Combining (10.29), (10.19) with the simple Hölder estimate

$$\|\mathbf{u}\|_p^2\geqslant\|\mathbf{u}\|_2^{\frac{8-4\eta}{3}}\|\mathbf{u}\|_1^{-\frac{2-4\eta}{3}},$$

we have

$$\partial_s\|\mathbf{u}\|_2^2\leqslant -c_\eta K^{-\frac{2}{3}\eta}\|\mathbf{u}\|_2^{\frac{8-4\eta}{3}}\|\mathbf{u}\|_1^{-\frac{2-4\eta}{3}},$$

i.e.,

$$\partial_s\|\mathbf{u}\|_2\leqslant -c_\eta K^{-\frac{2}{3}\eta}\|\mathbf{u}\|_2^{\frac{5-4\eta}{3}}\|\mathbf{u}\|_1^{-\frac{2-4\eta}{3}},\ \text{ so }\ -\frac{1}{\|\mathbf{u}(t)\|_2^{\frac{2-4\eta}{3}}}\leqslant -c_\eta K^{-\frac{2}{3}\eta}t\|\mathbf{u}\|_1^{-\frac{2-4\eta}{3}}$$

since $\|\mathbf{u}\|_1$ is decreasing. Thus

$$\|\mathbf{u}(t)\|_2\leqslant\Big(\frac{1}{(c_\eta K^{-\frac{2}{3}\eta}t)^{3/2}}\Big)^{1-6\eta}\|\mathbf{u}_0\|_1,$$

and by duality

$$\|\mathbf{u}(t)\|_\infty\leqslant\Big(\frac{1}{(c_\eta K^{-\frac{2}{3}\eta}t)^{3}}\Big)^{1-6\eta}\|\mathbf{u}_0\|_1.$$

Thus, after interpolation we have proved (10.18). □

Now we apply Proposition 10.4 to our case.

Corollary 10.6. *Fix $S\leqslant T$ and set $\mathcal{A}(s)=\widetilde{\mathcal{A}}(\mathbf{x}(s))$ as defined in* (9.1)*. On the set $\mathcal{G}$, the coefficients of $\mathcal{A}(s)=\mathcal{B}(s)+\mathcal{W}(s)$ satisfy* (10.16) *and* (10.17) *with the constant $b=cN^{-2\xi}$. Consequently, the solution to*

$$\partial_t\mathbf{u}(t)=-\mathcal{A}(t)\mathbf{u}(t)$$

satisfies

$$\|\mathbf{u}(s)\|_q\leqslant C(p,q,\eta)\Big(\frac{N^{2\xi+\frac{2}{3}\eta}}{s}\Big)^{(\frac{3}{p}-\frac{3}{q})(1-6\eta)}\|\mathbf{u}(0)\|_p,\qquad 0<s\leqslant S,\quad 1\leqslant p\leqslant q\leqslant\infty. \tag{10.22}$$

Proof. From the estimates on B and W proved in (10.14, 10.15), we have proved the estimates on the kernel elements in Lemma 10.4 with $b=N^{-C\xi}$. Thus (10.22) directly follows from (10.18). □

10.5 Second time cutoff

Now we specialize the observable $Q(\mathbf{x})$ to be of the form (7.25). Thus Q depends only on variables with indices in $\Lambda\subset[\![1,K^\zeta]\!]$ and $|\Lambda|=m$ with m a finite fixed number. Its derivative is bounded by

$$|\partial_jQ(\mathbf{x})|=\bigg\|j^{1/3}(\partial_jO)\bigg(\Big(i^{1/3}(x_i-\alpha_i)\Big)_{i\in\Lambda}\bigg)\bigg\|_\infty\leqslant K^{\zeta/3},\qquad j\in\Lambda. \tag{10.23}$$

With the help of Corollary 10.6, we can reduce the upper limit of the time integration in (10.11) from $T \sim K^{1/3}$ to $\widetilde{T} \sim K^{1/6}$. More precisely, using the $L^1 \to L^\infty$ bound of (10.22) with the choice $\eta = \xi$, the integration from $\widetilde{T}$ to T in (10.11) is bounded by

$$\begin{aligned}
&\frac{1}{2}\int_{\widetilde{T}}^{T} \mathrm{d}S \int \omega(\mathrm{d}\mathbf{x}) \sum_{a,b=1}^{K} \partial_b Q(\mathbf{x}) \mathbb{E}^{\mathbf{x}}\Big[\mathcal{G}\ \partial_a h_0(\mathbf{x}(S))|\ v_a^b(S,\mathbf{x}(\cdot))\Big] \\
&\qquad \leqslant CN^{\xi}|\Lambda|K^{\zeta/3} \max_{b\leqslant K^\zeta} \int_{\widetilde{T}}^{T} \sum_{a=1}^{K} \mathbb{E}^{\omega}\Big[\mathcal{G}\ |\partial_a h_0(\mathbf{x}(S))|\ v_a^b(S,\mathbf{x}(\cdot))\Big]\mathrm{d}S \\
&\qquad \leqslant CN^{C\xi}K^{(1+\zeta)/3} \max_{b} \int_{\widetilde{T}}^{T} \|\mathbf{v}^b(S,\mathbf{x}(\cdot))\|_\infty \mathrm{d}S \\
&\qquad \leqslant CN^{C\xi}K^{(1+\zeta)/3} \int_{\widetilde{T}}^{T} S^{-3(1-6\eta)}\mathrm{d}S \\
&\qquad \leqslant CN^{C\xi}K^{(1+\zeta)/3}\widetilde{T}^{-2}, \qquad (10.24)
\end{aligned}$$

where we also used (10.23) and (10.4) together with the fact that, on the set $\mathcal{G}$, $\mathbf{x}(S)$ satisfies (10.3).

Choosing

$$\widetilde{T} = CN^{C_5\xi}K^{(1+\zeta)/6} \qquad (10.25)$$

with a sufficiently large constant C_5, we conclude from (10.11) and (10.24) that

$$|\langle Q; h_0\rangle_\omega| \leqslant CN^{\xi}K^{\zeta} \max_{b\leqslant K^\zeta} \int_0^{\widetilde{T}} \sum_{a=1}^{K} \mathbb{E}^{\omega}\Big[\mathcal{G}\ |\partial_a h_0(\mathbf{x}(S))|\ v_a^b(S,\mathbf{x}(\cdot))\Big]\mathrm{d}S + O(N^{-C\xi}) \qquad (10.26)$$

with the special choice of Q from (7.25).

10.6 A space-time decay estimate and completion of the proof of Theorem 7.1

Using (10.4), the first term in (10.26) is estimated by

$$CN^{\xi}K^{\zeta}\sum_a \int_0^{\widetilde{T}} \mathbb{E}^{\omega}\mathcal{G}|\partial_a h_0(\mathbf{x}(S))| v_a^b(S,\mathbf{x}(\cdot))\mathrm{d}S \leqslant \sum_a \frac{N^{C\xi}K^{1/3}}{K+1-a}\int_0^{\widetilde{T}} \mathbb{E}^{\omega}\big[\mathcal{G}v_a^b(S)\big]\mathrm{d}S. \qquad (10.27)$$

The last term can be estimated using a new space-time decay estimate for the equation (10.12). Roughly speaking, the energy method asserts that the total dissipation is bounded by the initial L^2 norm. We will apply this idea to the vector $\{v_i^{\alpha/2}\}$, see (10.30), and combine it with a new Sobolev inequality to obtain a a control on the time integral of a weighted L^∞ norm in terms of the L^α norm (the weight comes from the fact that the dissipative term is inhomogenous in space). More precisely, we have the following estimate.

Lemma 10.7. *Consider $\mathcal{A}(s) = \widetilde{\mathcal{A}}(\mathbf{x}(s))$ as defined in* (9.1)*. Suppose that the coefficients $\mathcal{A}(s) = \mathcal{B}(s)+\mathcal{W}(s)$ satisfy* (10.16) *and* (10.17) *with a constant b. Then for any exponent $\alpha > 1$ there is a constant C_α such that the solution to*

$$\partial_t \mathbf{v}(t) = -\mathcal{A}(t)\mathbf{v}(t)$$

satisfies, for any integer $1 \leqslant M \leqslant K$ and for any positive time $t > 0$,

$$\int_0^t |v_M(s)|^\alpha \mathrm{d}s \leqslant C_\alpha M^{-2/3}C^{\sqrt{\log M}}b^{-1}(t+1)\|\mathbf{v}(0)\|_\alpha^\alpha. \qquad (10.28)$$

Proof. With some positive constant $c_\alpha > 0$ we have the following estimate for the solution $\mathbf{v} = \mathbf{v}(t)$:

$$\begin{aligned}\partial_t \|\mathbf{v}\|_\alpha^\alpha = \alpha \sum_i |v_i|^{\alpha-1}(\operatorname{sgn} v_i)\partial_t v_i &\leqslant -\alpha \sum_{i,j} |v_i|^{\alpha-1}(\operatorname{sgn} v_i)\mathcal{B}_{ij}[v_i - v_j] \\ &= -\frac{\alpha}{2}\sum_{i,j}\big[|v_i|^{\alpha-1}(\operatorname{sgn} v_i) - |v_j|^{\alpha-1}(\operatorname{sgn} v_j)\big]\mathcal{B}_{ij}[v_i - v_j], \\ &\leqslant -c_\alpha \sum_{i,j}\mathcal{B}_{ij}\big[|v_i|^{\alpha/2} - |v_j|^{\alpha/2}\big]^2, \end{aligned} \tag{10.29}$$

where we dropped the potential term $\alpha\sum_i |v_i|^{\alpha-1}(\operatorname{sgn} v_i)W_i v_i \geqslant 0$ and used the symmetry of $\mathcal{B}_{ij}$ in the first step. In the second step we used $\mathcal{B}_{ij} \geqslant 0$ and the straighforward calculus inequality

$$\big[|x|^{\alpha-1}\operatorname{sgn}(x) - |y|^{\alpha-1}\operatorname{sgn}(y)\big](x-y) \geqslant c_\alpha'\big[|x|^{\alpha/2} - |y|^{\alpha/2}\big]^2, \qquad x, y \in \mathbb{R},$$

with some $c_\alpha' > 0$. Integrating (10.29) from 0 to any $t > 0$ we thus have

$$\int_0^t \sum_{i,j}\mathcal{B}_{ij}(s)\big[|v_i(s)|^{\alpha/2} - |v_j(s)|^{\alpha/2}\big]^2 \leqslant \|\mathbf{v}(0)\|_\alpha^\alpha - \|\mathbf{v}(t)\|_\alpha^\alpha \leqslant \|\mathbf{v}(0)\|_\alpha^\alpha. \tag{10.30}$$

Using the lower bound on the coefficients of $\mathcal{B}(s)$, we get

$$\int_0^t \sum_{i\neq j}\frac{\big[|v_i(s)|^{\alpha/2} - |v_j(s)|^{\alpha/2}\big]^2}{(i^{2/3} - j^{2/3})^2}\mathrm{d}s \leqslant Cb^{-1}\|\mathbf{v}(0)\|_\alpha^\alpha.$$

Now we formulate another Sobolev-type inequality which will be proved in Appendix B.

Theorem 10.8. *There is a constant $C > 0$ such that for any $M \in I$ and $\mathbf{u} \in \mathbb{C}^M$ we have*

$$|u_M|^2 \leqslant M^{-2/3}C^{\sqrt{\log M}}\left[\sum_{i\neq j=1}^M \frac{(u_i - u_j)^2}{(i^{2/3} - j^{2/3})^2} + \sum_{i=1}^M |u_i|^2\right].$$

The factor $C^{\sqrt{\log M}}$ is probably an artifact of our proof. The factor $M^{-2/3}$ is optimal as we can take $u_M = 1$ and $u_j = 0$ for all other $j \neq M$.

Using Theorem 10.8 with the choice $u_i = |v_i|^{\alpha/2}$, we have for any $M \leqslant K$,

$$\int_0^t |v_M(s)|^\alpha \mathrm{d}s \leqslant M^{-2/3}C^{\sqrt{\log M}}b^{-1}\Big[\|\mathbf{v}(0)\|_\alpha^\alpha + \int_0^t \|\mathbf{v}(s)\|_\alpha^\alpha \mathrm{d}s\Big] \leqslant M^{-2/3}C^{\sqrt{\log M}}b^{-1}(t+1)\|\mathbf{v}(0)\|_\alpha^\alpha,$$

where in the last step we used that the L^α norm does not increase in time by (10.29). This completes the proof of Lemma 10.7. □

Proof of Theorem 7.1. On the set $\mathcal{G}$, the coefficients of $\mathcal{A}(s) = \mathcal{B}(s) + \mathcal{W}(s)$ satisfy the bounds (10.16) and (10.17) with the constant $b = cN^{-2\xi}$. Using a Hölder inequality

$$\int_0^{\widetilde{T}} |v_M(s)|\mathrm{d}s \leqslant \widetilde{T}^{1-\frac{1}{\alpha}}\Big(\int_0^{\widetilde{T}} |v_M(s)|^\alpha \mathrm{d}s\Big)^{1/\alpha}, \tag{10.31}$$

and then (10.28), with the choice of $t = \widetilde{T}$ from (10.25), we can complete the bound (10.27):

$$\sum_a \frac{N^{C\xi}K^{1/3}}{K+1-a}\int_0^{\widetilde{T}} \mathbb{E}^\omega\big[\mathcal{G}v_a^b(S)\big]\mathrm{d}S \leqslant C_\alpha \sum_a \frac{N^{C\xi}K^{1/3}K^{(1+\zeta)/6}}{K+1-a}\, a^{-\frac{2}{3\alpha}} \leqslant C_\alpha N^{C\xi}K^{-\frac{4-(3+\zeta)\alpha}{6\alpha}}. \tag{10.32}$$

Combining (10.26), (10.27) and (10.32) we get

$$|\langle O; h_0\rangle_\omega| \leqslant C_\alpha N^{C\xi}K^{-\frac{4-(3+\zeta)\alpha}{6\alpha}} + O(N^{-C\xi})$$

with the special choice of Q from (7.25). For any $\zeta < 1$ there exists an $\alpha > 1$ such that the exponent of K is negative. Then, with a sufficiently small ξ (depending on ζ, α and δ) we obtain (7.24) and this completes the proof of Theorem 7.1. □

Proof of Theorem 3.4. We follow the proof of Theorem 7.1, but instead of $Q(\mathbf{x})$ and $h_0(\mathbf{x})$ we use the simple observables $q(x_i)$, $f(x_j)$ depending on a single coordinate. Then the analogue of (10.24) gives a bound $N^{C\xi}\widetilde{T}^{-2}\|q'\|_\infty\|f'\|_\infty$ and (10.26) reads

$$\langle q(x_i); f(x_j)\rangle \leqslant N^{C\xi}\|q'\|_\infty\|f'\|_\infty \int_0^{\widetilde{T}} \mathbb{E}^\omega\big[\mathcal{G}\, v_j^i(S)\big]\,\mathrm{d}S \leqslant C_\alpha N^{C\xi}\|q'\|_\infty\|f'\|_\infty\Big[j^{-2/3\alpha}\widetilde{T}^{1-\frac{1}{\alpha}}\widetilde{T}^{\frac{1}{\alpha}} + \widetilde{T}^{-2}\Big],$$

where in the last step we used (10.28) with (10.31) as above and an inequality similar to (10.24) with $\partial_a h_0 = \delta_{0i}$ (notice that the factor $K^{(1+\zeta)/3}$ is not needed now.) Choosing α very close to 1, we can replace $j^{-2/3\alpha}$ with $j^{-2/3}$ at the expense of increasing the constant C in the exponent of $N^{C\xi}$. Optimizing these two estimates yields the choice $\widetilde{T} = j^{2/9}$ and thus

$$\langle q(x_i); f(x_j)\rangle \leqslant C_\alpha N^{C\xi} j^{-4/9}\|q'\|_\infty\|f'\|_\infty.$$

Taking into account the rescaling explained in Section 7.1, which results in the additional factor $N^{4/3}$ due to the derivatives, this proves (3.15) in Theorem 3.4. □

A Proof of lemmas 6.21, 6.22 and 6.23

A.1 Proof of Lemma 6.21.

Let $\varepsilon > 0$ be fixed and arbitrarily small as in the statement of the lemma and in the definition of $\Omega_{\rm Int}^{(N)}(3a/4+\varepsilon,\tau)$. In this proof, the notation $A(N,k,j,z) \lesssim B(N,k,j,z)$ means that there is an absolute constant $c > 0$ depending only on V such that for any element $z \in \Omega_{\rm Int}^{(N)}(a,\tau)$ one has $|A(N,k,j,z)| \leqslant c\,|B(N,k,j,z)|$. In the same way, $A \sim B$ means $c^{-1}|B| \leqslant |A| \leqslant c|B|$ for some $c > 0$ depending only on V.

For any fixed E with $A \leqslant E \leqslant B$, we define the index j such that $\gamma_j = \min\{\gamma_i : \gamma_i \geqslant E\}$. For notational simplicity, we assume without loss of generality that $j \leqslant N/2$ and $A = 0$. Note that

$$E \sim (j/N)^{2/3} \tag{A.1}$$

when $E \geqslant N^{-2/3}$, from the definition of j, and $|E - j| \leqslant CN^{-2/3}j^{-1/3}$. Moreover, we will often use the fact that, as a consequence of $z \in \Omega_{\rm Int}^{(N)}(3a/4+\varepsilon,\tau)$, we have

$$\eta \geqslant N^{-1+\frac{3a}{4}+\varepsilon}E^{-1/2} \sim N^{-\frac{2}{3}+\frac{3a}{4}+\varepsilon}j^{-\frac{1}{3}}. \tag{A.2}$$

We define

$$\Delta_k := \frac{1}{z-\lambda_k} - \frac{1}{z-\mathbb{E}(\lambda_k)} = \frac{\lambda_k - \mathbb{E}(\lambda_k)}{(z-\mathbb{E}(\lambda_k))(z-\lambda_k)}. \tag{A.3}$$

Then for any fixed b (we will choose b close to a, $a < b < a + \varepsilon/10$) we have

$$\frac{1}{N^2}\mathrm{Var}\left(\sum_{k=1}^{N} \frac{1}{z-\lambda_k}\right) \lesssim \Sigma_{\mathrm{Int}} + \Sigma_{\mathrm{ExtRight}} + \Sigma_{\mathrm{ExtLeft}}$$

where

$$\Sigma_{\mathrm{Int}} = \frac{1}{N^2}\mathbb{E}\left(\sum_{|k-j|\leqslant N^b} \Delta_k\right)^2,\ \Sigma_{\mathrm{ExtRight}} = \frac{1}{N^2}\mathbb{E}\left(\sum_{k>j+N^b} \Delta_k\right)^2,\ \Sigma_{\mathrm{ExtLeft}} = \frac{1}{N^2}\mathbb{E}\left(\sum_{k<j-N^b} \Delta_k\right)^2,$$

(some of these summations may be empty). We use the improved concentration result, $|\lambda_k - \mathbb{E}(\lambda_k)| \leqslant N^{-\frac{2}{3}+\frac{a}{2}+\varepsilon'}k^{-\frac{1}{3}}$ with very high probability for any ε' (Proposition 6.2) to bound the numerator in (A.3), and the rigidity at scale a to bound the denominator, which allows us to replace λ_k and α_k with γ_k, and $z = E + \mathrm{i}\eta$ with $\gamma_j + \mathrm{i}\eta$ whenever $|j-k| \geqslant N^b$, since in this regime the error is much smaller than $|\gamma_k - \gamma_j|$. To see this statement more precisely, first observe that we can assume that $k < 2N/3$, i.e., $\widehat{k} \sim k$; the large k regime is trivial since $j \leqslant N/2$. We make a distinction between two cases.

- We first assume that $k \geqslant j/2$. Notice that

$$|E-\lambda_k| \geqslant |\gamma_j - \lambda_k| - CN^{-2/3} \geqslant c|\gamma_j - \gamma_k| - CN^{-2/3} - CN^{-\frac{2}{3}+a+\varepsilon'}(\widehat{k})^{-1/3} \tag{A.4}$$

  from the choice of j and from the rigidity bound for λ_k with any $\varepsilon' > 0$. Since $|j-k| \geqslant N^b$, we have $|\gamma_j - \gamma_k| \geqslant cN^{-2/3+b}[\max(j,k)]^{-1/3}$. Using that $b > a$, one can choose $\varepsilon' > 0$ such that $|\gamma_j - \gamma_k|$ in (A.4) dominates the two error terms, for $k \geqslant j/2$.

- Suppose now that $k \leqslant j/2$, then (A.4) can be improved by noticing that

$$\lambda_k \leqslant \lambda_{j/2} \leqslant \gamma_{j/2} + CN^{-\frac{2}{3}+a+\varepsilon'}j^{-1/3}$$

  (using rigidity for $\lambda_{j/2}$), thus we can use

$$|E-\lambda_k| \geqslant \gamma_j - \gamma_{j/2} - CN^{-2/3} - CN^{-\frac{2}{3}+a+\varepsilon'}j^{-1/3} \geqslant c|\gamma_j - \gamma_k| - CN^{-2/3} - CN^{-\frac{2}{3}+a+\varepsilon'}j^{-1/3} \tag{A.5}$$

  instead of (A.4). Since $k \leqslant j/2$, we have $|\gamma_j - \gamma_k| \geqslant cN^{-2/3+b}j^{-1/3}$, which is larger than the error term in (A.5).

To summarize, we proved the following estimates:

$$\Sigma_{\mathrm{Int}} \lesssim \frac{1}{N^2}\left(\sum_{|k-j|\leqslant N^b} \frac{N^{-\frac{2}{3}+\frac{b}{2}}k^{-\frac{1}{3}}}{\eta^2}\right)^2,$$

$$\Sigma_{\mathrm{ExtRight}} \lesssim \frac{1}{N^2}\left(\sum_{k\geqslant j+N^b} \frac{N^{-\frac{2}{3}+\frac{b}{2}}k^{-\frac{1}{3}}}{\eta^2 + (\gamma_k - \gamma_j)^2}\right)^2,$$

$$\Sigma_{\mathrm{ExtLeft}} \lesssim \frac{1}{N^2}\left(\sum_{1\leqslant k\leqslant j-N^b} \frac{N^{-\frac{2}{3}+\frac{b}{2}}k^{-\frac{1}{3}}}{\eta^2 + (\gamma_k - \gamma_j)^2}\right)^2.$$

Sum over internal points. We first consider Σ_{Int}. This is smaller than

$$\frac{N^{-\frac{4}{3}+b}}{N^2\eta^4}\left(\sum_{\ell=\max(1,j-N^b)}^{j+N^b}\ell^{-\frac{1}{3}}\right)^2 \lesssim N^{-\frac{10}{3}+b}\eta^{-4}(N^{2b}j^{-2/3}\mathbb{1}_{j\geqslant N^b}+N^{\frac{4b}{3}}\mathbb{1}_{j\leqslant N^b}) \lesssim N^{-\frac{10}{3}+3b}\eta^{-4}j^{-2/3}.$$

This last term is, as expected, smaller than $N^{-1+\frac{3a}{4}}\eta^{-1}\max(E^{\frac{1}{2}},\eta^{\frac{1}{2}})$ which holds for the following reasons.

- Case $\eta\leqslant E$. The desired inequality is $N^{-\frac{7}{3}+3b-\frac{3a}{4}}\lesssim\sqrt{E}j^{\frac{2}{3}}\eta^3$.

  As $E\sim(j/N)^{2/3}$, the desired inequality is $\eta\gg N^{-\frac{2}{3}}j^{-\frac{1}{3}}N^{b-\frac{a}{4}}$. This holds because $z\in\Omega_{\text{Int}}^{(N)}(3a/4+\varepsilon,\tau)$, hence $\eta\gtrsim N^{-1+\frac{3a}{4}+\varepsilon}E^{-\frac{1}{2}}\sim N^{-\frac{2}{3}+\frac{3a}{4}+\varepsilon}j^{-\frac{1}{3}}$, and $b\leqslant a+\varepsilon/10$.

- Case $\eta\geqslant E$. The desired inequality is $\eta^{7/2}\gg N^{-\frac{7}{3}+3b-\frac{3a}{4}}j^{-2/3}$. We distinguish two cases. For large j, namely for $j\gg N^{b-\frac{a}{4}}$, from $\eta\geqslant E$ we have

$$\eta^{\frac{7}{2}}\geqslant E^{\frac{7}{2}}=\left(\frac{j}{N}\right)^{\frac{7}{3}}\gg N^{-\frac{7}{3}+3b-\frac{3a}{4}}j^{-2/3}.$$

  On the other hand, from (A.2) we have

$$\eta^{\frac{7}{2}}\geqslant N^{-\frac{7}{3}}j^{-\frac{7}{6}}N^{\frac{7}{2}(\frac{3a}{4}+\varepsilon)}\gg N^{-\frac{7}{3}+3b-\frac{3a}{4}}j^{-2/3}$$

  whenever $j\ll N^{(\frac{27}{4}a-6b)+7\varepsilon}$. As $(\frac{27}{4}a-6b)+7\varepsilon>b-\frac{a}{4}$, we have either $j\gg N^{b-\frac{a}{4}}$ or $j\ll N^{(\frac{27}{4}a-6b)+7\varepsilon}$, so in any case we have proved the expected result.

Sum over external points on the right. We now consider Σ_{ExtRight}. Note that when $\ell\geqslant 0$, $\gamma_{j+\ell}-\gamma_j=\ell N^{-\frac{2}{3}}j^{-\frac{1}{3}}\mathbb{1}_{\ell\leqslant j}+\left(\frac{\ell}{N}\right)^{\frac{2}{3}}\mathbb{1}_{\ell>j}$, consequently

$$\Sigma_{\text{ExtRight}}\leqslant\Sigma_1+\Sigma_2,\ \Sigma_1=\frac{1}{N^2}\left(\sum_{N^b\leqslant\ell\leqslant j}\frac{N^{-\frac{2}{3}+\frac{b}{2}}j^{-\frac{1}{3}}}{\eta^2+\ell^2N^{-\frac{4}{3}}j^{-\frac{2}{3}}}\right)^2,\ \Sigma_2=\frac{1}{N^2}\left(\sum_{\ell\geqslant j}\frac{N^{-\frac{2}{3}+\frac{b}{2}}\ell^{-\frac{1}{3}}}{\eta^2+\left(\frac{\ell}{N}\right)^{\frac{4}{3}}}\right)^2.$$

We first consider Σ_1. This summation is non-empty if $j\geqslant N^b\geqslant N^a$.

- In the case $E\leqslant\eta\leqslant\tau$, we have

$$\Sigma_1\leqslant\frac{1}{N^2}\left(\sum_{N^b\leqslant\ell\leqslant j}\frac{N^{-\frac{2}{3}+\frac{b}{2}}j^{-\frac{1}{3}}}{\eta^2}\right)^2=\frac{N^{-\frac{10}{3}}}{\eta^4}N^bj^{\frac{4}{3}}\ll\frac{N^{\frac{3a}{4}}}{N\eta}\sqrt{\eta},$$

  where the last step holds because $\eta^{\frac{7}{2}}\geqslant E^{\frac{7}{2}}\gg\left(\frac{j}{N}\right)^{\frac{7}{3}}N^{b-\frac{3a}{4}}j^{-1}$, where the last inequality follows from (A.1) and the fact that $j\geqslant N^b$.

- If $\eta\leqslant E$, we first consider the case $\eta\leqslant N^{-\frac{2}{3}+a}j^{-\frac{1}{3}}$. The following holds (using (A.1))

$$\Sigma_1\leqslant\frac{1}{N^2}\left(\sum_{N^b\leqslant\ell\leqslant j}\frac{N^{-2/3+\frac{b}{2}}j^{-\frac{1}{3}}}{\ell^2N^{-\frac{4}{3}}j^{-\frac{2}{3}}}\right)^2=\left(\frac{j}{N}\right)^{\frac{2}{3}}N^{-b}\ll\frac{N^{\frac{3a}{4}}}{N\eta}\sqrt{E},$$

  because $\eta\leqslant N^{-\frac{2}{3}+a}j^{-\frac{1}{3}}\ll N^{-\frac{2}{3}+\frac{3a}{4}+b}j^{-\frac{1}{3}}$.

- In the last possible case $N^{-\frac{2}{3}+a}j^{-\frac{1}{3}} \leqslant \eta \leqslant E$, we have

$$\Sigma_1 \leqslant \frac{1}{N^2}\left(\sum_{N^b \leqslant \ell \leqslant \eta N^{\frac{2}{3}} j^{-\frac{1}{3}}} \frac{N^{-\frac{2}{3}+\frac{b}{2}} j^{-\frac{1}{3}}}{\eta^2}\right)^2 + \frac{1}{N^2}\left(\sum_{\eta N^{\frac{2}{3}} j^{-\frac{1}{3}} \leqslant \ell \leqslant j} \frac{N^{-\frac{2}{3}+\frac{b}{2}} j^{-\frac{1}{3}}}{\ell^2 N^{-\frac{4}{3}} j^{-\frac{2}{3}}}\right)^2 = \frac{N^b}{N^2\eta^2} \ll \frac{N^{\frac{3a}{4}}}{N\eta}\sqrt{E}$$

  because $\eta \geqslant N^{-\frac{2}{3}+a}j^{-\frac{1}{3}} \gg N^{-\frac{2}{3}+b-\frac{3a}{4}}j^{-\frac{1}{3}}$ and we used (A.1).

We now consider the term Σ_2.

- If $\eta \leqslant E$, we have

$$\Sigma_2 \leqslant \frac{1}{N^2}\left(\sum_{\ell \geqslant j} \frac{N^{-\frac{2}{3}+\frac{b}{2}} \ell^{-\frac{1}{3}}}{(\ell/N)^{4/3}}\right)^2 = N^{-\frac{2}{3}+b} j^{-\frac{4}{3}} \ll \frac{N^{\frac{3a}{4}}}{N\eta}\left(\frac{j}{N}\right)^{\frac{1}{3}} = \frac{N^{\frac{3a}{4}}}{N\eta}\sqrt{E},$$

  where in the last inequality we used $\eta \leqslant E \sim (j/N)^{2/3}$ and $j \gg N^{b-\frac{3a}{4}}$, this last relation holds because on $\{\eta \leqslant E\} \cap \Omega_{\rm Int}^{(N)}(3a/4+\varepsilon,\tau)$ we have $j \geqslant N^{\frac{3a}{4}+\varepsilon}$.

- If $\eta \geqslant E$, we have

$$\Sigma_2 \leqslant \frac{1}{N^2}\left(\sum_{j \leqslant \ell \leqslant N\eta^{\frac{3}{2}}} \frac{N^{-\frac{2}{3}+\frac{b}{2}} \ell^{-\frac{1}{3}}}{\eta^2}\right)^2 + \frac{1}{N^2}\left(\sum_{N\eta^{\frac{3}{2}} \leqslant \ell \leqslant N} \frac{N^{-\frac{2}{3}+\frac{b}{2}} \ell^{-\frac{1}{3}}}{(\ell/N)^{\frac{4}{3}}}\right)^2 = \frac{N^b}{N^2\eta^2} \ll \frac{N^{\frac{3a}{4}}}{N\eta}\sqrt{\eta},$$

  where in the last step we used that on the domain $\{\eta \geqslant E\} \cap \Omega_{\rm Int}^{(N)}$, we have $\eta \geqslant N^{-\frac{2}{3}+\frac{a}{2}+\frac{2}{3}\varepsilon}$.

Sum over external points on the left. We now consider $\Sigma_{\rm ExtLeft}$, which is non-trivial only for $j \geqslant N^b \geqslant N^a$. Beginning similarly to the previous paragraph, we can write

$$\Sigma_{\rm ExtLeft} \leqslant \widetilde{\Sigma}_1 + \widetilde{\Sigma}_2,\ \widetilde{\Sigma}_1 = \frac{1}{N^2}\left(\sum_{1 \leqslant k \leqslant \frac{j}{2}} \frac{N^{-\frac{2}{3}} k^{-\frac{1}{3}} N^{\frac{b}{2}}}{\eta^2 + \left(\frac{j}{N}\right)^{\frac{4}{3}}}\right)^2,\ \widetilde{\Sigma}_2 = \frac{1}{N^2}\left(\sum_{\frac{j}{2} \leqslant k \leqslant j-N^b} \frac{N^{-\frac{2}{3}} j^{-\frac{1}{3}} N^{\frac{b}{2}}}{\eta^2 + (j-k)^2 N^{-\frac{4}{3}} j^{-\frac{2}{3}}}\right)^2.$$

A calculation yields

$$\widetilde{\Sigma}_1 = N^{-2+b} E^2 \min(\eta^{-4}, E^{-4}) \ll \frac{N^{\frac{3a}{4}}}{N\eta} \max(E^{\frac{1}{2}}, \eta^{\frac{1}{2}}),$$

where in the last step we used the following.

- If $\eta \geqslant E$, then the desired inequality is $N^{-1+b-\frac{3a}{4}} E^2 \leqslant \eta^{7/2}$ which follows from $N^{-1+b-\frac{3a}{4}} \leqslant E^{3/2} \sim j/N$, which holds since $j \geqslant N^a \geqslant N^{b-\frac{3a}{4}}$.

- If $\eta \leqslant E$, the desired relation is $N^{-2+b}E^{-2} \leqslant \frac{N^{\frac{3a}{4}}}{N\eta}\sqrt{E}$ which again follows from $N^{-1+b-\frac{3a}{4}} \leqslant E^{3/2} \sim j/N$ as before, since $j \geqslant N^a \geqslant N^{b-\frac{3a}{4}}$.

We now consider the $\widetilde{\Sigma}_2$ term.

- If $\eta \leqslant E$ and $N^{-\frac{2}{3}+\frac{3a}{4}+\varepsilon}j^{-\frac{1}{3}} \leqslant \eta \leqslant N^{-\frac{2}{3}+a}j^{-\frac{1}{3}}$, we have $\widetilde{\Sigma}_2 = N^{-\frac{2}{3}}j^{\frac{2}{3}}N^{-b}$, so the desired result $\widetilde{\Sigma}_2 \ll \frac{N^{\frac{3a}{4}}}{N\eta}\sqrt{E} \sim \frac{N^{\frac{3a}{4}}}{N\eta}\left(\frac{j}{N}\right)^{\frac{1}{3}}$ is equivalent to $N^{-\frac{2}{3}}j^{\frac{2}{3}}N^{-b} \ll \frac{N^{\frac{3a}{4}}}{N\eta}\left(\frac{j}{N}\right)^{\frac{1}{3}}$, i.e., $\eta \ll N^{b+\frac{3a}{4}+\varepsilon}N^{-2/3}j^{-1/3}$, which obviously holds by the assumption $\eta \leqslant N^{-\frac{2}{3}+a}j^{-\frac{1}{3}}$.
- If $\eta \leqslant E$ and $N^{-\frac{2}{3}+a}j^{-\frac{1}{3}} \leqslant \eta$, $\widetilde{\Sigma}_2$ is bounded by

$$\frac{1}{N^2}\left(\sum_{\frac{j}{2}\leqslant \ell\leqslant j-\eta N^{2/3}j^{1/3}} \frac{N^{-\frac{2}{3}}j^{-\frac{1}{3}}N^{\frac{b}{2}}}{(j-\ell)^2 N^{-\frac{4}{3}}j^{-\frac{2}{3}}}\right)^2 + \frac{1}{N^2}\left(\sum_{j-\eta N^{2/3}j^{1/3}\leqslant \ell\leqslant j-N^b} \frac{N^{-\frac{2}{3}}j^{-\frac{1}{3}}N^{\frac{b}{2}}}{\eta^2}\right)^2 = \frac{N^b}{N^2\eta^2} \ll \frac{N^{\frac{3a}{4}}}{N\eta}\sqrt{E},$$

where in the last step we used (A.1) and that $\eta \gg N^{-\frac{2}{3}}j^{-\frac{1}{3}}N^{b-\frac{3a}{4}}$ from (A.2).
- If $E \leqslant \eta \leqslant \tau$, we also have $\widetilde{\Sigma}_2 \leqslant \frac{N^b}{N^2\eta^2}$ which is properly bounded, exactly as we proved it for the proof of Σ_2 on the domain $\{\eta \geqslant E\}$.

A.2 Proof of Lemma 6.22

Let $d > 2a/3$ and $b > 3a/4$. On $\Omega^{(N)}_{\rm Ext}(d,\tau)$, we have $\eta \geqslant c\kappa_E$, so we want to prove that uniformly in $z \in \Omega^{(N)}_{\rm Ext}(d,\tau)$ we have

$$\frac{1}{N^2}\,\mathrm{Var}\left(\sum_{i=1}^N \frac{1}{z-\lambda_i}\right) \ll \frac{1}{N\eta}\eta^{1/2}.$$

We know that

$$\frac{1}{N^2}\left|\mathrm{Var}\left(\sum_{i=1}^N \frac{1}{z-\lambda_i}\right)\right| \lesssim \Sigma_{\rm Int} + \Sigma_{\rm Ext},$$

$$\Sigma_{\rm Int} = \frac{1}{N^2}\mathbb{E}^\mu\left|\sum_{i\leqslant N^b}\left(\frac{1}{z-\lambda_i} - \frac{1}{z-\mathbb{E}^\mu(\lambda_i)}\right)\right|^2 \lesssim \frac{1}{N^2}\left(\sum_{i\leqslant N^b}\frac{N^{-\frac{2}{3}+\frac{a}{2}}i^{-\frac{1}{3}}}{\eta^2}\right)^2,$$

$$\Sigma_{\rm Ext} = \frac{1}{N^2}\mathbb{E}^\mu\left|\sum_{i> N^b}\left(\frac{1}{z-\lambda_i} - \frac{1}{z-\mathbb{E}^\mu(\lambda_i)}\right)\right|^2 \lesssim \frac{1}{N^2}\left(\sum_{i> N^b}\frac{N^{-\frac{2}{3}+\frac{a}{2}}i^{-\frac{1}{3}}}{\eta^2+(|E-A|+\left(\frac{i}{N}\right)^{\frac{2}{3}})^2}\right)^2,$$

where we used concentration at scale $a/2$ that follows from the conditions of Lemma 6.22 using Proposition 6.2. In the last equation, we additionally used accuracy at scale $3a/4$ for $i \geqslant N^b$ to obtain that $|z-\lambda_i| \sim |z-\mathbb{E}(\lambda_i)| \sim |z-\gamma_i|$. The term $\Sigma_{\rm Int}$ is therefore easily bounded by $N^{-\frac{10}{3}+a+\frac{4}{3}b}\eta^{-4}$, and

$$\Sigma_{\rm Ext} \leqslant N^{-\frac{10}{3}+a}\left(\sum_{i>1}\frac{i^{-\frac{1}{3}}}{\eta^2+\left(\frac{i}{N}\right)^{\frac{4}{3}}}\right)^2 \lesssim \frac{N^{-2+a}}{\eta^2}.$$

This concludes the proof because, for $\eta \geqslant N^{-\frac{2}{3}+d}$, $d > 2a/3$, we have both

$$\frac{N^{-\frac{10}{3}+2a}}{\eta^4} \leqslant \frac{1}{N\eta}\eta^{1/2} \quad \text{and} \quad \frac{N^{-2+a}}{\eta^2} \leqslant \frac{1}{N\eta}\eta^{1/2}.$$

.

A.3 Proof of Lemma 6.23

Let $b > 3a/4$. We begin with the bound on m_N':

$$\left|\frac{1}{N}m_N'(z)\right| \leqslant \frac{1}{N^2}\sum_{i\leqslant N^b}\mathbb{E}\left(\frac{1}{|z-\lambda_i|^2}\right)+\frac{1}{N^2}\sum_{i>N^b}\mathbb{E}\left(\frac{1}{|z-\lambda_i|^2}\right).$$

If $N^{-\frac{2}{3}+a+\varepsilon} > |z-A| > N^{-\frac{2}{3}+d}$ and $z \in \Omega^{(N)}(d,s,\tau)$, then the contributions of both terms are easily bounded by

$$\frac{1}{N^2}N^b(N^{-\frac{2}{3}+s})^{-2}+\frac{1}{N^2}\sum_{i\geqslant 1}\frac{1}{\left(|z-A|+\left(\frac{i}{N}\right)^{\frac{2}{3}}\right)^2} \leqslant N^{-\frac{2}{3}+b-2s}+N^{-1}|z-A|^{-\frac{1}{2}}. \tag{A.6}$$

If $|z-A| > N^{-\frac{2}{3}+a+\varepsilon}$, then by rigidity at scale a we can use the second term in (A.6) to estimate all indices $i \geqslant 1$. By choosing $b = 3a/4+\varepsilon$, this gives the expected result (6.47).

We now bound the variance term, in the same way as in the previous subsection:

$$\frac{1}{N^2}\left|\mathrm{Var}\left(\sum_{i=1}^{N}\frac{1}{z-\lambda_i}\right)\right| \lesssim \Sigma_{\mathrm{Int}}\mathbb{1}_{N^{-\frac{2}{3}+a+\varepsilon}>|z-A|>N^{-\frac{2}{3}+d}}+\Sigma_{\mathrm{Ext}},$$

$$\Sigma_{\mathrm{Int}} = \frac{1}{N^2}\mathbb{E}\left|\sum_{i\leqslant N^b}\left(\frac{1}{z-\lambda_i}-\frac{1}{z-\mathbb{E}^{\mu}(\lambda_i)}\right)\right|^2 \lesssim \frac{1}{N^2}\left(\sum_{i\leqslant N^b}\frac{N^{-\frac{2}{3}+\frac{a}{2}}i^{-\frac{1}{3}}}{\eta^2}\right)^2,$$

$$\Sigma_{\mathrm{Ext}} = \frac{1}{N^2}\mathbb{E}\left|\sum_{i>N^b}\left(\frac{1}{z-\lambda_i}-\frac{1}{z-\mathbb{E}^{\mu}(\lambda_i)}\right)\right|^2 \lesssim \frac{1}{N^2}\left(\sum_{i>1}\frac{N^{-\frac{2}{3}+\frac{a}{2}}i^{-\frac{1}{3}}}{(|z-A|+\left(\frac{i}{N}\right)^{\frac{2}{3}})^2}\right)^2.$$

The announced bounds then follow by a computation of the above terms.

B Two Sobolev-type inequalities

In this section we prove two Sobolev type inequalities. The first one has a discrete and continuous version, the second one is valid only in the discrete setup.

Proof of Proposition 10.5. We start with the proof of (10.20). We recall the representation formula for fractional powers of the Laplacian: for any $0 < \alpha < 2$ function f on $\mathbb{R}$ we have

$$\langle f, |p|^{\alpha} f\rangle = C(\alpha)\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(f(x)-f(y))^2}{|x-y|^{1+\alpha}}\mathrm{d}x\mathrm{d}y \tag{B.1}$$

with some explicit constant $C(\alpha)$, where $|p| := \sqrt{-\Delta}$.

In order to bring the left hand side of (10.20) into the form similar to (B.1), we estimate, for $0 < x < y$,

$$y^{2/3}-x^{2/3} = (3/2)\int_x^y s^{-1/3}\mathrm{d}s \leqslant C(y-x)(xy)^{-1/6}$$

(for $y-x\leqslant x$ we have $x\sim y$ and it follows directly, for $y-x\geqslant x$, i.e., $y\geqslant 2x$, and $y-x\sim y$ we get $\int_x^y s^{-1/3}ds\sim y^{2/3}\leqslant (y-x)(xy)^{-1/6}$). Thus to prove (10.20), it is sufficient to show that

$$\int_0^\infty\int_0^\infty \frac{(f(x)-f(y))^2}{|x-y|^{2-\eta}}(xy)^q\mathrm{d}x\mathrm{d}y\geqslant c_\eta\Big(\int_0^\infty |f(x)|^p\mathrm{d}x\Big)^{2/p},\qquad p=\frac{3}{1+\eta},\quad q:=\frac{1}{3}-\frac{\eta}{6},\tag{B.2}$$

holds for any function supported on $[0,\infty]$.

Now we symmetrize f, i.e., define $\widetilde{f}$ on $\mathbb{R}$ such that $\widetilde{f}(x)=f(x)$ for $x>0$ and $\widetilde{f}(x)=f(-x)$ for $x<0$. Then

$$2\int_0^\infty\int_0^\infty \frac{(f(x)-f(y))^2}{|x-y|^{2-\eta}}|xy|^q\mathrm{d}x\mathrm{d}y=\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(\widetilde{f}(x)-\widetilde{f}(y))^2}{|x-y|^{2-\eta}}|xy|^q\mathrm{d}x\mathrm{d}y-2\int_0^\infty\int_0^\infty\frac{(f(x)-f(y))^2}{|x+y|^{2-\eta}}|xy|^q\mathrm{d}x\mathrm{d}y$$

$$\geqslant\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(\widetilde{f}(x)-\widetilde{f}(y))^2}{|x-y|^{2-\eta}}|xy|^q\mathrm{d}x\mathrm{d}y-2\int_0^\infty\int_0^\infty\frac{(f(x)-f(y))^2}{|x-y|^{2-\eta}}|xy|^q\mathrm{d}x\mathrm{d}y,$$

where we used that $|x+y|\geqslant|x-y|$ for positive numbers. Thus

$$\int_0^\infty\int_0^\infty \frac{(f(x)-f(y))^2}{|x-y|^{2-\eta}}(xy)^q\mathrm{d}x\mathrm{d}y\geqslant\frac{1}{4}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(\widetilde{f}(x)-\widetilde{f}(y))^2}{|x-y|^{2-\eta}}|xy|^q\mathrm{d}x\mathrm{d}y.$$

Since

$$\int_0^\infty |f(x)|^p dx=\frac{1}{2}\int_{\mathbb{R}}|\widetilde{f}(x)|^p\mathrm{d}x,$$

the estimate (B.2) would follow from

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(\widetilde{f}(x)-\widetilde{f}(y))^2}{|x-y|^{2-\eta}}|xy|^q\mathrm{d}x\mathrm{d}y\geqslant c'_\eta\Big(\int_{\mathbb{R}}|\widetilde{f}(x)|^p\mathrm{d}x\Big)^{2/p},\qquad p:=\frac{3}{1+\eta}.\tag{B.3}$$

Setting

$$\phi(x):=|x|^q,$$

(B.3) is equivalent to

$$\int_{-\infty}^\infty\int_{-\infty}^\infty\frac{(f(x)-f(y))^2}{|x-y|^{2-\eta}}\phi(x)\phi(y)\mathrm{d}x\mathrm{d}y\geqslant c_\eta\Big(\int_{\mathbb{R}}|f(x)|^p\mathrm{d}x\Big)^{2/p}\tag{B.4}$$

for any function f on $\mathbb{R}$ (for simplicitly we dropped the tilde in f and the prime in c_η).

We have

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(f(x)-f(y))^2}{|x-y|^{2-\eta}}\phi(x)\phi(y)\mathrm{d}x\mathrm{d}y=\lim_{\varepsilon\to0}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(f(x)-f(y))^2}{|x-y|^{2-\eta}+\varepsilon}\phi(x)\phi(y)\mathrm{d}x\mathrm{d}y$$

$$=\lim_{\varepsilon\to0}\Bigg[2\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{f(x)^2}{|x-y|^{2-\eta}+\varepsilon}\phi(x)\phi(y)\mathrm{d}x\mathrm{d}y-2\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\phi(x)f(x)\phi(y)f(y)}{|x-y|^{2-\eta}+\varepsilon}\mathrm{d}x\mathrm{d}y\Bigg]$$

$$=\lim_{\varepsilon\to0}\Bigg[2\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(\phi(x)f(x))^2}{|x-y|^{2-\eta}+\varepsilon}\mathrm{d}x\mathrm{d}y-2\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\phi(x)f(x)\phi(y)f(y)}{|x-y|^{2-\eta}+\varepsilon}\mathrm{d}x\mathrm{d}y\Bigg]$$

$$+\lim_{\varepsilon\to 0} 2\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{f(x)^2}{|x-y|^{2-\eta}+\varepsilon}(\phi(y)-\phi(x))\phi(x)\mathrm{d}x\mathrm{d}y$$

$$=\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(\phi(x)f(x)-\phi(y)f(y))^2}{|x-y|^{2-\eta}+\varepsilon}\mathrm{d}x\mathrm{d}y+\lim_{\varepsilon\to 0} 2\int_{\mathbb{R}}|f(x)|^2\Big[\int_{\mathbb{R}}\frac{\phi(y)-\phi(x)}{|x-y|^{2-\eta}+\varepsilon}dy\Big]\phi(x)\mathrm{d}x.$$

The first term is

$$(\phi f, |p|^{1-\eta}\phi f).$$

Since f is symmetric, we can assume $x>0$ in computing the second term:

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\frac{\phi(y)-x^q}{|x-y|^{2-\eta}+\varepsilon}\mathrm{d}y=\lim_{\varepsilon\to 0}\int_0^\infty\frac{y^q-x^q}{|x-y|^{2-\eta}+\varepsilon}\mathrm{d}y+\lim_{\varepsilon\to 0}\int_{-\infty}^0\frac{(-y)^q-x^q}{|x-y|^{2-\eta}+\varepsilon}\mathrm{d}y$$

$$=\lim_{\varepsilon\to 0}\int_0^\infty\frac{y^q-x^q}{|x-y|^{2-\eta}+\varepsilon}\mathrm{d}y+\lim_{\varepsilon\to 0}\int_0^\infty\frac{y^q-x^q}{|x+y|^{2-\eta}+\varepsilon}\mathrm{d}y$$

$$=x^{q-1+\eta}\Big[\lim_{\varepsilon\to 0}\int_0^\infty\frac{u^q-1}{|u-1|^{2-\eta}+\varepsilon}\mathrm{d}u+\lim_{\varepsilon\to 0}\int_0^\infty\frac{u^q-1}{|u+1|^{2-\eta}+\varepsilon}\mathrm{d}u\Big]=C_0(\eta)x^{q-1+\eta}.$$

We need that $C_0(\eta)>0$ for small η. Since C_0 is clearly continuous, it is sufficient to show that $C_0(0)>0$. This can be seen by the $v=1/u$ substitution for $u\geqslant 1$

$$\int_0^\infty\frac{u^q-1}{|u\pm 1|^2}\mathrm{d}u=\int_0^1\frac{u^q-1}{|u\pm 1|^2}\mathrm{d}u+\int_0^1\frac{(1/v)^q-1}{|(1/v)\pm 1|^2}\frac{\mathrm{d}v}{v^2}=\int_0^1\frac{u^q+u^{-q}-2}{|u\pm 1|^2}\mathrm{d}u>0$$

since $u^q+u^{-q}\geqslant 2$. (What we really used about the weight function ϕ is that $\frac{1}{2}(\phi(a)+\phi(1/a))\geqslant\phi(1)$ for any $a>0$.) Once $C_0(0)>0$, we can choose a sufficiently small $\eta>0$ so that $C_0(\eta)>0$ as well. From now on we fix such a small η.

In summary, we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{(f(x)-f(y))^2}{|x-y|^{2-\eta}}\phi(x)\phi(y)\mathrm{d}x\mathrm{d}y=\langle\phi f,|p|^{1-\eta}\phi f\rangle+C_0(\eta)\int_{\mathbb{R}}\frac{|f(x)|^2}{|x|^{1-2q-\eta}}\mathrm{d}x$$

$$=\langle\phi f,|p|^{1-\eta}\phi f\rangle+C_0(\eta)\int_{\mathbb{R}}\frac{|\phi(x)f(x)|^2}{|x|^{1-\eta}}\mathrm{d}x.$$

So the positive term can be dropped and in order to prove (B.4), we need to prove

$$\langle f\phi,|p|^{1-\eta}\phi f\rangle\geqslant c_\eta\Big(\int_{\mathbb{R}}|f|^p\Big)^{2/p}.$$

Denote $g=|p|^{\frac{1}{2}(1-\eta)}|x|^q f$, (recall $q=\frac{1}{3}-\frac{\eta}{6}$), we need to prove that

$$\|g\|_2\geqslant c_\eta\big\||x|^{-q}|p|^{-\frac{1}{2}(1-\eta)}g\big\|_p.$$

Recall the weighted Hardy-Littlewood-Sobolev inequality [52] in n-dimensions

$$\Big\||x|^{-q}\int|x-y|^{-a}g(y)\mathrm{d}y\Big\|_p\leqslant C\|g\|_r,\quad \frac{1}{r}+\frac{a+q}{n}=1+\frac{1}{p},\quad 0\leqslant q<n/p,\quad 0<a<n.$$

In our case, $a=(1+\eta)/2, r=2, n=1$, and all conditions are satisfied if we take $0<\eta<1$. This completes the proof of the continuous part of Proposition 10.5. Part (ii), the discrete version (10.21), follows from (10.20) by linear interpolation exactly as in the proof of Proposition B.2 in [25]. □

Proof of Theorem 10.8. Take $1 \leqslant \ell \leqslant j \leqslant M$ and estimate

$$\begin{aligned}
|u_j|^2 \leqslant & 2\left|\frac{1}{\ell}\sum_{i=j-\ell}^{j-1}(u_j-u_i)\right|^2 + 2\left|\frac{1}{\ell}\sum_{i=j-\ell}^{j-1}u_i\right|^2 \\
\leqslant & \frac{2}{\ell^2}\left(\sum_{i=j-\ell}^{j-1}\frac{(u_j-u_i)^2}{(j^{2/3}-i^{2/3})^2}\right)\left(\sum_{i=j-\ell}^{j-1}(j^{2/3}-i^{2/3})^2\right) + \frac{2}{\ell}\sum_{i=j-\ell}^{j-1}|u_i|^2 \\
\leqslant & C\ell j^{-2/3}\sum_{i=j-\ell}^{j-1}\frac{(u_j-u_i)^2}{(j^{2/3}-i^{2/3})^2} + \frac{2}{\ell}\sum_{i=j-\ell}^{j-1}|u_i|^2,
\end{aligned}$$

where we performed the summation

$$\sum_{i=j-\ell}^{j-1}(j^{2/3}-i^{2/3})^2 \leqslant C\ell^3 j^{-2/3}.$$

We will apply this whenever $j \geqslant M/2$, so $j^{-2/3}$ will be replaced by $CM^{-2/3}$. So for any $1 \leqslant \ell \leqslant j \leqslant M$, $j \geqslant M/2$, we have

$$|u_j|^2 \leqslant C_0\ell M^{-2/3}\sum_{i=j-\ell}^{j-1}\frac{(u_j-u_i)^2}{(j^{2/3}-i^{2/3})^2} + \frac{2}{\ell}\sum_{i=j-\ell}^{j-1}|u_i|^2, \tag{B.5}$$

with some fixed constant C_0.

Choose an increasing sequence $\ell_1 \leqslant \ell_2 \leqslant \ldots \leqslant \ell_{n+1}$ such that $\ell_{j+1} \geqslant 2\ell_j$ and n such that $\ell_{n+1} \leqslant M/2$. We use (B.5) for $j = M$ and $\ell = \ell_1$:

$$|u_M|^2 \leqslant C_0\ell_1 M^{-2/3}\sum_{i=M-\ell_1}^{M-1}\frac{(u_M-u_i)^2}{(M^{2/3}-i^{2/3})^2} + \frac{2}{\ell_1}\sum_{j=M-\ell_1}^{m-1}|u_j|^2 \leqslant C_0\ell_1 M^{-2/3}D + \frac{2}{\ell_1}\sum_{j=M-\ell_1}^{M-1}|u_j|^2, \tag{B.6}$$

where we denote

$$D := \sum_{i\neq j=1}^{M}\frac{(u_i-u_j)^2}{(i^{2/3}-j^{2/3})^2}.$$

Now for each u_j in the last sum we can use (B.5) again, but now with $\ell = \ell_2$

$$\begin{aligned}
\frac{2}{\ell_1}\sum_{j=M-\ell_1}^{M-1}|u_j|^2 \leqslant & \frac{2C_0\ell_2}{\ell_1}M^{-2/3}\sum_{j=M-\ell_1}^{M-1}\sum_{i=j-\ell_2}^{j-1}\frac{(u_j-u_i)^2}{(j^{2/3}-i^{2/3})^2} + \frac{4}{\ell_2\ell_1}\sum_{j=M-\ell_1}^{M-1}\sum_{i=j-\ell_2}^{j-1}|u_i|^2 \\
\leqslant & C_0 M^{-2/3}D\frac{2\ell_2}{\ell_1} + \frac{4}{\ell_2}\sum_{i=M-\ell_1-\ell_2}^{M-1}|u_i|^2.
\end{aligned}$$

Combining with (B.6) we get

$$|u_M|^2 \leqslant C_0 M^{-2/3}D\Big(\ell_1 + \frac{2\ell_2}{\ell_1}\Big) + \frac{4}{\ell_2}\sum_{j=M-\ell_1-\ell_2}^{M-1}|u_j|^2.$$

Continuing this procedure, after n steps we get

$$|u_M|^2 \leqslant C_0 M^{-2/3} D\Big(\ell_1 + \frac{2\ell_2}{\ell_1} + \frac{4\ell_3}{\ell_2} + \cdots + \frac{2^n \ell_{n+1}}{\ell_n}\Big) + \frac{2^{n+1}}{\ell_{n+1}} \sum_{j=M-\ell_1-\cdots-\ell_{n+1}}^{M-1} |u_j|^2$$

and the recursion works since up to the last step the running index j satisfied $j \geqslant M - \ell_1 - \cdots - \ell_n \geqslant M - \ell_{n+1} \geqslant M/2$ by the choice of n. Optimizing the choices we have

$$|u_M|^2 \leqslant 2^n C_0 M^{-2/3} D \ell_{n+1}^{\frac{1}{n}} + \frac{2^{n+1}}{\ell_{n+1}} \sum_j |u_j|^2.$$

We can choose $\ell_{n+1} = M^{2/3}$ and $n = \sqrt{\log M}$, then

$$|u_M|^2 \leqslant C M^{-2/3}\Big(M^{\frac{2}{3n}} + 2^n\Big)(D + \|u\|_2) \leqslant M^{-2/3} C^{\sqrt{\log M}}\Big(D + \sum_{i=1}^M |u_i|^2\Big),$$

which completes the proof. □

C Proof of Lemma 10.1

Let $M = N^{C\xi}$ with a constant $C > C_3$ (from the definition of $\mathcal{G}$). We define

$$V_{\mathbf{y}}^*(x) := V\big(xN^{-2/3}\big) - \frac{2}{N} \sum_{k > K+M} \log|x - y_k|, \qquad x \in J_{\mathbf{y}} = (-\infty, y_+], \tag{C.1}$$

i.e., we write

$$V_{\mathbf{y}}(x) = V_{\mathbf{y}}^*(x) - \frac{2}{N} \sum_{k=K+1}^{K+M} \log|x - y_k|,$$

where we split the external points into two sets. The nearby external points (with indices $K+1 \leqslant k \leqslant K+M$) are kept explicitly, while the far away points, y_k, $k > K+M$ are kept together with the potential in $V_{\mathbf{y}}^*$ because there is a cancellation between them to explore. The proof of the following lemma on the derivative of $V_{\mathbf{y}}^*$ is postponed to the end of this section.

Lemma C.1. *For any $\mathbf{y} \in \mathcal{R}_K(\xi)$ and $M = N^{C\xi}$ with a large constant C, we have*

$$[V_{\mathbf{y}}^*]'(x) = N^{-2/3} \int_0^{[(K+M)/N]^{2/3}} \frac{\varrho(y)\mathrm{d}y}{xN^{-2/3} - y} + O\Big(\frac{N^{-1+\xi}}{|(K+M)^{2/3} - x|}\Big), \qquad x \in [0, y_+] \tag{C.2}$$

and

$$\big|[V_{\mathbf{y}}^*]'(x)\big| \leqslant CN^{-1+2\xi} K^{1/3}, \qquad x \in [-N^{C\xi}, y_+]. \tag{C.3}$$

Here we assume that the density ϱ satisfies (2.8).

From the definitions (7.23), (C.1) we claim that

$$|\partial_j h_0(\mathbf{x})| \leqslant \sum_{k=K+2}^{K+M} \Big[\frac{1}{|x_j - y_k|} - \frac{1}{|x_j - \widetilde{y}_k|}\Big] + N\big|[V^*_{\mathbf{y}}]'(x_j) - [\widetilde{V}^*_{\widetilde{\mathbf{y}}}]'(x_j)\big| \tag{C.4}$$

$$\leqslant \frac{CN^{C\xi}K^{1/3}}{K+1-j} + N\big|[V^*_{\mathbf{y}}]'(x_j) - [\widetilde{V}^*_{\widetilde{\mathbf{y}}}]'(x_j)\big|.$$

Notice that the summation over k starts from $k = K+2$, this is because the boundary terms $|x_j - y_{K+1}|^{-1} = |x_j - \widetilde{y}_{K+1}|^{-1}$, present both in $V_{\mathbf{y}}(x_j)$ and $\widetilde{V}_{\widetilde{\mathbf{y}}}(x_j)$, cancel out. Using $|x_j - y_k| \geqslant |y_{K+2} - y_{K+1}| \geqslant N^{-\xi}K^{1/3}$ by the definition of $\mathbf{y} \in \mathcal{R}^{\#}$, each term in the summation is bounded by $N^{\xi}K^{1/3}$. So its contribution is at most $CMN^{\xi}K^{1/3} \leqslant CN^{C\xi}K^{1/3}$. We will use this bound for $j \geqslant K - N^{C\xi}$. For $j \leqslant K - N^{C\xi}$, we use $|x_j - y_k| \geqslant |x_j - y_{K+1}| \geqslant c|\gamma_j - \gamma_K| \geqslant cK^{-1/3}|K-j|$ and this gives the estimate on the first term in (C.4).

For the second term in (C.4) we use (C.2) to have

$$N\big|[V^*_{\mathbf{y}}]'(x_j) - [\widetilde{V}^*_{\widetilde{\mathbf{y}}}]'(x_j)\big| \leqslant N^{1/3}\int_0^{[(K+M)/N]^{2/3}} \frac{\big[\varrho(y) - \widetilde{\varrho}(y)\big]\mathrm{d}y}{x_j N^{-2/3} - y} + O\Big(\frac{N^{\xi}}{|(K+M)^{2/3} - x_j|}\Big). \tag{C.5}$$

Notice that $x_j \sim j^{2/3}$ with a precision smaller than $N^{C_3\xi}j^{-1/3}$ since

$$|x_j - j^{2/3}| \leqslant |x_j - \gamma_j| + |\gamma_j - j^{2/3}| \leqslant N^{C_3\xi}j^{-1/3} + j^{4/3}N^{-2/3} \leqslant N^{C_3\xi}j^{-1/3}, \qquad j \leqslant K,$$

by the definition of $\mathcal{G}$, by (7.3) and (10.1). Thus we have

$$|(K+M)^{2/3} - x_j| \geqslant |(K+M)^{2/3} - j^{2/3}| - N^{C_3\xi}j^{-1/3} \geqslant K^{-1/3}|K+M-j| - N^{C_3\xi}j^{-1/3} \geqslant cK^{-1/3}|K+M-j|$$

using that $M = N^{C\xi} \gg N^{C_3\xi}$. Thus the error term in (C.5) is bounded by the r.h.s. of (10.4).

Finally, in the main term of (C.5) we use the asymptotics (7.15). The density $\varrho(y) - \widetilde{\varrho}(y)$ is a C^1-function of size of order $y^{3/2} \leqslant C(K/N)$ on the integration domain. Thus a simple analysis, similar to the proof of (C.9) in the Appendix shows that

$$N^{1/3}\int_0^{[(K+M)/N]^{2/3}} \frac{\big[\varrho(y) - \widetilde{\varrho}(y)\big]\mathrm{d}y}{x_j N^{-2/3} - y} \leqslant CN^{1/3}(K/N)(\log N)$$

which is smaller than the r.h.s. of (10.4) by (10.1). This proves (10.4). The proof of (10.5) trivially follows from (10.4). This completes the proof of Lemma 10.1.

Proof of Lemma C.1. For any fixed $x \in [0, y_+]$ we have

$$\frac{1}{2}\big(V^*_{\mathbf{y}}(x)\big)' = \frac{1}{2}N^{-2/3}V'(xN^{-2/3}) - \frac{1}{N}\sum_{k>K+M}\frac{1}{x - y_k} \tag{C.6}$$

$$= N^{-2/3}\int \frac{\varrho(y)\mathrm{d}y}{xN^{-2/3} - y} - \frac{1}{N}\sum_{k>K+M}\frac{1}{x - y_k}, \qquad x \in [0, y_+],$$

where we have used the equation (7.2). Thanks to rigidity, $\mathbf{y} \in \mathcal{R}_K(\xi)$, we can replace y_k's with γ_k's at an error

$$|\mathcal{E}_1| := \bigg|\frac{1}{N}\sum_{k>K+M}\Big[\frac{1}{x - y_k} - \frac{1}{x - \gamma_k}\Big]\bigg| \leqslant \frac{1}{N}\sum_{k>K+M}\frac{|y_k - \gamma_k|}{(x - \gamma_k)^2}$$

$$\leqslant \frac{CN^{\xi}}{N}\sum_{k>K+M}\frac{1}{\widehat{k}^{1/3}(x - \widehat{k}^{2/3})^2} \leqslant \frac{CN^{-1+\xi}}{|(K+M)^{2/3} - x|} \tag{C.7}$$

where we also used that for any $x \leqslant y_+ \leqslant \gamma_{K+1} + CN^{\xi}(K+1)^{-1/3}$ and $k \geqslant K+M$ we have $\gamma_k - x \geqslant c(\gamma_k - \gamma_{K+1})$ since $\gamma_k - \gamma_{K+1} \geqslant cMK^{-1/3}$ is larger than the rigidity error $CN^{\xi}K^{-1/3}$. For the purpose of the estimates, we can thus replace γ_k with $\widehat{k}^{2/3}$. The last step in (C.7) is a simple estimate.

After replacement, we have to control

$$\frac{1}{N}\sum_{k>K+M}\frac{1}{x-\gamma_k} = N^{-2/3}\frac{1}{N}\sum_{k>K+M}\frac{1}{xN^{-2/3}-\Gamma_k} = N^{-2/3}\int_Q \frac{\varrho(y)\mathrm{d}y}{xN^{-2/3}-y} + \mathcal{E}_2$$

with $Q := [N^{-2/3}\gamma_{K+M+1}, B]$. The error $\mathcal{E}_2$ can be written as

$$\mathcal{E}_2 = N^{-2/3}\sum_{k>K+M}\int_{\gamma_k/N^{2/3}}^{\gamma_{k+1}/N^{2/3}}\left[\frac{1}{xN^{-2/3}-y} - \frac{1}{xN^{-2/3}-\gamma_k N^{-2/3}}\right]\varrho(y)\mathrm{d}y$$

using

$$\int_{\gamma_k/N^{2/3}}^{\gamma_{k+1}/N^{2/3}} \varrho = 1/N.$$

Thus for $x \in [0, y_+]$ the error is bounded by

$$|\mathcal{E}_2| \leqslant \frac{CN^{\xi}}{N^{5/3}}\sum_{k>K+M}\Big(\frac{1}{\widehat{k}^{1/3}N^{2/3}}\Big)\frac{1}{[xN^{-2/3}-(\widehat{k}/N)^{2/3}]^2} \leqslant \frac{CN^{-1+\xi}}{|(K+M)^{2/3}-x|}$$

using $|y - N^{-2/3}\gamma_k| \leqslant CN^{\xi}\widehat{k}^{-1/3}N^{-2/3}$ for $y \in [\gamma_k/N^{2/3}, \gamma_{k+1}/N^{2/3}]$ and that $\gamma_k/N^{2/3} \sim (k/N)^{2/3}$. The calculation in the last line is the same as in (C.7). Thus

$$\frac{1}{2}\big(V_{\mathbf{y}}^*(x)\big)' = N^{-2/3}\int_0^{\gamma_{K+M+1}/N^{2/3}}\frac{\varrho(y)\mathrm{d}y}{xN^{-2/3}-y} + O\Big(\frac{N^{-1+\xi}}{|(K+M)^{2/3}-x|}\Big), \qquad x \in [0, y_+]. \tag{C.8}$$

Moreover, we have

$$N^{-2/3}\left|\int_{[(K+M)/N]^{2/3}}^{\gamma_{K+M+1}/N^{2/3}}\frac{\varrho(y)\mathrm{d}y}{xN^{-2/3}-y}\right| \leqslant \frac{C}{|(K+M)^{2/3}-x|}\int_{[(K+M)/N]^{2/3}}^{\gamma_{K+M+1}/N^{2/3}}\varrho(y)\mathrm{d}y = O\Big(\frac{N^{-1+\xi}}{|(K+M)^{2/3}-x|}\Big).$$

Here we used that $|x - N^{2/3}y|$ is comparable with $|x-(K+M)^{2/3}|$ and $\varrho(y) \leqslant C\sqrt{y} \leqslant C(K/N)^{1/3}$ on the integration domain and that $|\gamma_{K+M+1}/N^{2/3} - [(K+M)/N]^{2/3}| \leqslant C[K/N]^{4/3}$, see (7.3). Finally we used (10.1). Thus from (C.8) we obtained (C.2).

To obtain the bound in (C.3), we first notice in the error term in (C.2) we have $|(K+M)^{2/3}-x| \geqslant |(K+M)^{2/3}-y_+| \geqslant |(K+M)^{2/3}-(K+1)^{2/3}| - CN^{\xi}K^{-1/3} \geqslant cK^{-1/3}M$ since $M \geqslant CN^{\xi}$. So this can be bounded by the r.h.s. of (C.3).

The singular integral in (C.2), up to logarithmic factors, is bounded by the size of ϱ on this interval, which is at most $C(K/N)^{1/3}$, so this term is also bounded by the r.h.s. of (C.3). More precisely, for any $0 \leqslant b < u < a$ and for density ϱ satisfying (2.8), we claim that

$$\Big|\int_b^a \frac{\varrho(y)\mathrm{d}y}{u-y}\Big| \leqslant C|u|^{1/2}\max\{\log|u-b|, \log|a-u|\}. \tag{C.9}$$

Since in our case, by the choice of M, $u := xN^{-2/3}$ and $a := [(K+M)/N]^{2/3}$ are separated by at least N^{-1}, we indeed get (with $b = 0$)

$$\left| N^{-2/3} \int_0^{[(K+M)/N]^{2/3}} \frac{\varrho(y)\mathrm{d}y}{xN^{-2/3} - y} \right| \leqslant CN^{-2/3}(K/N)^{1/3}(\log N).$$

Finally, we need to consider the case $x < 0$. The only difference from the proof for $x > 0$ is that the equilibrium relation (7.2) holds with an error term:

$$\frac{1}{2}V'(x) = \int \frac{\varrho(y)\mathrm{d}y}{x-y} + O(N^{-1/3+C\xi}), \qquad x \in [-N^{-\frac{2}{3}+C\xi}, 0],$$

that can be easily seen by comparing it with the $x = 0$ case and using that V is smooth and

$$\Big| \int \frac{\varrho(y)\mathrm{d}y}{x-y} - \int \frac{\varrho(y)\mathrm{d}y}{-y} \Big| \leqslant |x| \int \frac{\varrho(y)\mathrm{d}y}{(|x|+y)y} \leqslant C|x|^{1/2} \leqslant CN^{-1/3+C\xi}$$

by $\varrho(y) \leqslant C\sqrt{y}$. This error term in (C.6) yields an error of size $N^{-1+C\xi}$ in the final result, which is smaller than the r.h.s. of (C.3). We thus proved Lemma C.1. □

D Level repulsion for the local measure: Proof of Theorem 3.2

The proof in this section uses ideas similar to those in [8,25]. Before we start the actual proof of Theorem 3.2, we need some Lemmas. We first introduce an auxiliary measure which is a slightly modified version of the local equilibrium measures:

$$\sigma_0 := Z^*(y_{K+1} - x_K)^{-\beta}\sigma_{\mathbf{y}},$$

where Z^* is chosen for normalization. In other words, we drop the term $(y_{K+1} - x_K)^{\beta}$ from the measure $\sigma_{\mathbf{y}}$ in σ_0. We first prove estimates weaker than (3.8)-(3.9) for $\sigma_{\mathbf{y}}$ and σ_0.

Lemma D.1. *Let $\mathbf{y} \in \mathcal{R} = \mathcal{R}_K(\xi)$. We have for any $s > 0$*

$$\mathbb{P}^{\sigma_{\mathbf{y}}}(y_{K+1} - x_K \leqslant sK^{-1/3}) \leqslant CK^2 s, \tag{D.1}$$

$$\mathbb{P}^{\sigma_{\mathbf{y}}}(y_{K+1} - x_K \leqslant sK^{-1/3}) \leqslant CN^{C\xi}s + e^{-N^c}. \tag{D.2}$$

The very same estimates hold if $\sigma_{\mathbf{y}}$ is replaced with σ_0.

Proof. We set $y_+ := y_{K+1}$ and $y_- := y_+ - a$ with $a := N^{\xi}K^{-1/3}$. By $\mathbf{y} \in \mathcal{R}_K$ we know that

$$y_{K+1} \geqslant \gamma_{K+1} - N^{\xi}(K+1)^{-1/3} \geqslant cK^{2/3},$$

thus

$$0 < y_- < y_+, \qquad y_+, y_- \sim K^{2/3}.$$

We decompose the configurational space according to the number of the particles in $[y_-, y_+]$, which we denote by n. For any $0 \leqslant \varphi \leqslant c$ (with a small constant smaller than 1/2) we consider

$$
\begin{aligned}
Z_\varphi :=& \sum_{n=0}^{K} \int \cdots \int_{-\infty}^{y_-} \Big(\prod_{j=1}^{K-n} \mathrm{d}x_j \Big) \int \cdots \int_{y_-}^{y_+ - a\varphi} \Big(\prod_{j=K-n+1}^{K} \mathrm{d}x_j \Big) \Bigg[\prod_{\substack{i,j \in I \\ i<j}} (x_j - x_i)^\beta \Bigg] e^{-N\frac{\beta}{2} \sum_{j\in I} V_{\mathbf{y}}(x_j) - 2\beta \sum_{j \in I} \Theta(N^{-\xi} x_j)} \\
=& \sum_{n=0}^{K} (1-\varphi)^{n+\beta n(n-1)/2} \int \cdots \int_{-\infty}^{y_-} \Big(\prod_{j=1}^{K-n} \mathrm{d}w_j \Big) \int \cdots \int_{y_-}^{y_+} \Big(\prod_{j=K-n+1}^{K} \mathrm{d}w_j \Big) \\
& \times \Bigg[\prod_{i<j \leqslant K-n} (w_j - w_i)^\beta \Bigg] \Bigg[\prod_{K-n<i<j\leqslant K} (w_j - w_i)^\beta \Bigg] \Bigg[\prod_{i \leqslant K-n} \prod_{j=K-n+1}^{K} (y_- + (1-\varphi)(w_j - y_-) - w_i)^\beta \Bigg] \\
& \times e^{-N\frac{\beta}{2} \left[\sum_{j\leqslant K-n} V_{\mathbf{y}}(w_j) + \sum_{j>K-n} V_{\mathbf{y}}(y_- + (1-\varphi)(w_j - y_-)) \right] - 2\beta \sum_{j\leqslant K-n} \Theta\left(N^{-\xi} w_j\right)},
\end{aligned}
$$

where in the n particle sector we changed variables to

$$
w_j := x_j \quad \text{for} \quad j \leqslant K-n, \qquad w_j := y_- + (1-\varphi)^{-1}(x_j - y_-) \quad \text{for} \quad K-n+1 \leqslant j \leqslant K.
$$

We also exploited the fact that for $x_j \geqslant y_- \geqslant 0$ we have $\Theta(N^{-\xi} x_j) = 0$.

Now we compare Z_φ with $Z_{\varphi=0}$. We fix n and we work in each sector separately. The mixed interaction terms can be estimated by

$$
\big[y_- + (1-\varphi)(w_j - y_-) - w_i\big]^\beta \geqslant \big[(1-\varphi)(w_j - w_i)\big]^\beta \tag{D.3}
$$

for any $w_i \leqslant y_- \leqslant w_j$. To estimate the effect of the scaling in the potential term $V_{\mathbf{y}}$, we fix a parameter M with $CN^\xi \leqslant M \leqslant K$. For $j > K-n$, i.e $w_j \in [y_-, y_+]$, we write

$$
\begin{aligned}
e^{-N\frac{\beta}{2} V_{\mathbf{y}}(y_- + (1-\varphi)(w_j - y_-))} &= e^{-N\frac{\beta}{2} V_{\mathbf{y}}^*(y_- + (1-\varphi)(w_j - y_-))} \\
&\quad \times \prod_{K+1 \leqslant k \leqslant K+M} (y_k - y_- - (1-\varphi)(w_j - y_-))^\beta.
\end{aligned} \tag{D.4}
$$

with the definition

$$
V_{\mathbf{y}}^*(x) := V\big(xN^{-2/3}\big) - \frac{2}{N} \sum_{k>K+M} \log|x - y_k|, \qquad x \in [y_-, y_+]. \tag{D.5}
$$

Notice that the index k is always between 1 and N, so any limits of summations automatically include this condition as well.

For the potential $V_{\mathbf{y}}^*$ we have

$$
\Big| V_{\mathbf{y}}^*(y_- + (1-\varphi)(w_j - y_-)) - V_{\mathbf{y}}^*(w_j) \Big| \leqslant \max_{x \in [y_-, y_+]} \big| \big(V_{\mathbf{y}}^*(x)\big)' \big|\, a\varphi \leqslant CN^{-1+C\xi} \varphi, \tag{D.6}
$$

where we have used $|w_j - y_-| \leqslant a = N^\xi K^{-1/3}$ and $j > K-n$. The derivative of $V_{\mathbf{y}}^*$ will be estimated in (C.3). In summary, from (D.6) we have the lower bound

$$
e^{-N\frac{\beta}{2} V_{\mathbf{y}}^*(y_- + (1-\varphi)(w_j - y_-))} e^{N\frac{\beta}{2} V_{\mathbf{y}}^*(w_j)} \geqslant e^{-C\varphi N^{C\xi}}, \qquad j > K-n.
$$

For the other factors in (D.4), we use $y_k - y_- - (1-\varphi)(w_j - y_-) \geqslant (1-\varphi)(y_k - w_j)$ if $k \geqslant K+1$, thus

$$\prod_{K+1\leqslant k\leqslant K+M} (y_k - y_- - (1-\varphi)(w_j - y_-))^\beta \geqslant (1-\varphi)^{M\beta} \prod_{K+1\leqslant k\leqslant K+M} (y_k - w_j)^\beta.$$

Choose $M = N^{C\xi}$. After multiplying these estimates for all $j \in I$ we thus have the bound

$$Z_\varphi \geqslant \sum_{n=0}^{K} (1-\varphi)^{n+\beta n(n-1)/2+\beta n(K-n)+\beta n N^{C\xi}} e^{-C\varphi n N^{C\xi}} \int \cdots \int_{-\infty}^{y_-} \Big(\prod_{j=1}^{K-n} \mathrm{d}w_j \Big) \int \cdots \int_{y_-}^{y_+} \Big(\prod_{j=K-n+1}^{K} \mathrm{d}w_j \Big)$$
$$\times \left[\prod_{i<j\leqslant K} (w_j - w_i)^\beta \right] e^{-N\frac{\beta}{2}\sum_{j\leqslant K} V_{\mathbf{y}}(w_j) - 2\beta \sum_{j\leqslant K-n} \Theta\left(N^{-\xi} w_j\right)}.$$

Since $n \leqslant K$, we can estimate

$$(1-\varphi)^{n+\beta n(n-1)/2+\beta n(K-n)+\beta n N^{C\xi}} e^{-C\varphi n N^{C\xi}} \geqslant (1-\varphi)^{CK^2}, \tag{D.7}$$

and after bringing this factor out of the summation, the remaining sum is just $Z_{\varphi=0}$. We thus have

$$\frac{Z_\varphi}{Z_0} \geqslant (1-\varphi)^{CK^2}.$$

Now we choose $\varphi := sK^{-1/3}a^{-1} = sN^{-\xi}$. Therefore the $\sigma_{\mathbf{y}}$-probability of $y_{K+1} - x_K \geqslant sK^{-1/3} = a\varphi$ can be estimated by

$$\mathbb{P}^{\sigma_{\mathbf{y}}}(y_{K+1} - x_K \geqslant sK^{-1/3}) = \frac{Z_\varphi}{Z_0} \geqslant 1 - CsK^2.$$

This proves (D.1).

For the proof of (D.2), we first insert the characteristic function of the set

$$\mathcal{G}_0 := \big\{ \mathbf{x} \,:\, |x_j - \alpha_j| \leqslant CN^\xi j^{-1/3},\ j \in I \big\}.$$

into the integral defining Z_φ and denote the new quantity by $Z_\varphi^{\mathcal{G}}$. Clearly $Z_\varphi \geqslant Z_\varphi^{\mathcal{G}}$ and by the rigidity bound (3.7) we know that

$$\frac{Z_0 - Z_0^{\mathcal{G}}}{Z_0} = \mathbb{P}^{\sigma_{\mathbf{y}}}(\mathcal{G}_0^c) \leqslant Ce^{-N^c},$$

thus (with $\varphi = sN^{-\xi}$ as above)

$$\mathbb{P}^{\sigma_{\mathbf{y}}}(y_{K+1} - x_K \geqslant sK^{-1/3}) = \frac{Z_\varphi}{Z_0} \geqslant \frac{Z_\varphi^{\mathcal{G}}}{Z_0^{\mathcal{G}}}\big(1 - Ce^{-N^c}\big) \geqslant \frac{Z_\varphi^{\mathcal{G}}}{Z_0^{\mathcal{G}}} - Ce^{-N^c} \tag{D.8}$$

since $Z_\varphi^{\mathcal{G}} \leqslant Z_0^{\mathcal{G}}$.

To estimate $Z_\varphi^{\mathcal{G}}/Z_0^{\mathcal{G}}$, we follow the previous proof with two modifications. First we notice that the summation over n in the definition of Z_φ is restricted to $n \leqslant N^{2\xi}$ on the set $\mathcal{G}_0$, since no more than $N^{C\xi}$ particles can fall into the interval $[y_-, y_+]$ if they are approximately regularly spaced.

The other change concerns the estimate of the mixed terms (D.3) which will be improved to

$$\prod_{i\leqslant K-n}\prod_{j=K-n+1}^{K}\big[y_- + (1-\varphi)(w_j-y_-)-w_i\big]^\beta \geqslant \prod_{i\leqslant K-n}\prod_{j=K-n+1}^{K}\Big[(w_j-w_i)\big(1-\frac{\varphi(w_j-y_-)}{w_j-w_i}\big)\Big]^\beta$$

$$\geqslant \prod_{j=K-n+1}^{K}\left[\Big(1-\sum_{i\leqslant K-n}\frac{\varphi(w_j-y_-)}{w_j-w_i}\Big)_+\prod_{i\leqslant K-n}(w_j-w_i)\right]^\beta$$

for any $w_i \leqslant y_- \leqslant w_j$. Here we used that $\prod_i(1-a_i)\geqslant \big(1-\sum_i a_i\big)_+$ for any numbers $0\leqslant a_i\leqslant 1$. On the set $\mathcal{G}_0$ we have, by definitions of x_j and w_j, that

$$\sum_{i\leqslant K-n}\frac{\varphi(w_j-y_-)}{w_j-w_i}\leqslant 2\varphi\sum_{i\leqslant K-n}\frac{x_j-y_-}{x_j-x_i}.$$

Recall that $x_i\leqslant y_-\leqslant x_j$ and thus $\frac{x_j-y_-}{x_j-x_i}\leqslant 1$. For indices $i\leqslant K-CN^\xi$ with a sufficiently large C we can replace x_i with $\alpha_i\sim i^{2/3}$ with replacement error $CN^\xi i^{-1/3}$ which is smaller than $y_- -\alpha_i\leqslant x_j-x_i$. Together with $x_j-x_i\geqslant c(K^{2/3}-i^{2/3})$, we have

$$\sum_{i\leqslant K-n}\frac{\varphi(w_j-y_-)}{w_j-w_i}\leqslant C\varphi N^\xi + C\varphi N^\xi K^{-1/3}\sum_{i\leqslant K-CN^\xi}\frac{1}{K^{2/3}-i^{2/3}}\leqslant C\varphi N^\xi\log N. \tag{D.9}$$

The bound (D.9) will be used n-times, for all $j>K-n$.

Collecting these new estimates, instead of (D.7) we have the following prefactor depending on φ:

$$(1-\varphi)^{n+\beta n(n-1)/2+\beta nN^{C\xi}}(1-C\varphi N^\xi\log N)_+^n e^{-C\varphi nN^{C\xi}}\geqslant 1-CN^{C\xi}\varphi$$

using $n\leqslant N^{2\xi}$. This gives

$$\frac{Z_\varphi^{\mathcal{G}}}{Z_0^{\mathcal{G}}}\geqslant 1-CN^{C\xi}\varphi$$

which, together with (D.8) and with the choice $\varphi:=sK^{-1/3}a^{-1}=sN^{-\xi}$ gives (D.2).

The proof of (D.1)–(D.2) for σ_0 is very similar, just the $k=K+1$ factor is missing from (D.4) in case of $j=K$. This modification does not alter the basic estimates. This concludes the proof of Lemma D.1. □

Proof of Theorem 3.2. Recalling the definition of σ_0 and setting $X:=y_{K+1}-x_K$ for brevity, we have

$$\mathbb{P}^{\sigma_{\mathbf{y}}}[X\leqslant sK^{-1/3}]=\frac{\mathbb{E}^{\sigma_0}[\mathbb{1}(X\leqslant sK^{-1/3})X^\beta]}{\mathbb{E}^{\sigma_0}[X^\beta]}. \tag{D.10}$$

From (D.1) with σ_0 we have

$$\mathbb{E}^{\sigma_0}[\mathbb{1}(X\leqslant sK^{-1/3})X^\beta]\leqslant C(sK^{-1/3})^\beta K^2 s,$$

and with the choice $s=cK^{-2}$ in (D.1) (with σ_0) we also have

$$\mathbb{P}^{\sigma_0}\Big(X\geqslant\frac{c}{K^{1/3}K^2}\Big)\geqslant 1/2$$

with some positive constant c. This implies that

$$\mathbb{E}^{\sigma_0}[X^\beta] \geqslant \frac{1}{2}\left(\frac{c}{K^{1/3}K^2}\right)^\beta.$$

We have thus proved from (D.10) that

$$\mathbb{P}^{\sigma_{\mathbf{y}}}[X \leqslant sK^{-1/3}] \leqslant C(sK^{-1/3})^\beta K^2 s (K^{1/3}K^2)^\beta = C\left(K^2 s\right)^{\beta+1},$$

i.e., we obtained (3.8). The bound (3.9) follows similarly from (D.2) but with the choice $s = N^{-2C\xi}$ (where C is the constant in the exponent in (D.2)), and this completes the proof of Theorem 3.2. □

E Heuristics for the correlation decay in GUE

In this section we give a quick heuristic argument to justify the estimate (3.17), for a covariance w.r.t. the GUE measure. More precisely, for $V(x) = x^2/2, \beta = 2$, we have

$$\langle N^{2/3} i^{1/3}(\lambda_i - \gamma_i); N^{2/3} j^{1/3}(\lambda_j - \gamma_j)\rangle_\mu \sim \left(\frac{i}{j}\right)^{\frac{1}{3}},$$

for all $N^\delta \leqslant i \ll j \leqslant N/2$, where $\delta > 0$ is a small constant ($i \geqslant N^\delta$ is just a technical hypothesis allowing an easier use of Hermite polynomials asymptotics hereafter, the result should hold true without this condition). In the following, all but the first step can be made easily rigorous by following the method in [34]: formula (E.1) was easier in the context of diverging covariances (this divergence holds when $|i-j| \ll j$, i.e., $\theta_i < \delta$ with the notations of Corollary 2.3), for polynomially vanishing ones it would require a new rigorous argument. In this section $A \sim B$ means that $cB \leqslant A \leqslant c^{-1}B$ for some constant $c > 0$ independent of N.

(i) Let $\mathcal{N}(x) = |\{\ell : \lambda_\ell \leqslant x\}|$. Then asymptotic covariances of λ_i's are related to those of $\mathcal{N}(\gamma_i)'s$ (γ_i is the typical location, defined in (2.6)) by the following formula:

$$\langle N^{\frac{2}{3}} i^{\frac{1}{3}} \lambda_i, N^{\frac{2}{3}} j^{\frac{1}{3}} \lambda_j\rangle_\mu \sim \langle \mathcal{N}(\gamma_i), \mathcal{N}(\gamma_j)\rangle_\mu. \tag{E.1}$$

This relies on the idea that eigenvalues with close enough indexes move together, so for any x and y the events $\{\lambda_i - \gamma_i \leqslant x(\gamma_{i+1} - \gamma_i), \lambda_j - \gamma_j \leqslant y(\gamma_{j+1} - \gamma_j)\}$ and $\{\mathcal{N}(\gamma_i) - i \geqslant x, \mathcal{N}(\gamma_j) - j \geqslant y\}$ have very close probability. To be made rigorous, this step would require that for $\varepsilon > 0$ small enough and any $i_1 \in [\![i - N^\varepsilon, i + N^\varepsilon]\!]$, $j_1 \in [\![j - N^\varepsilon, j + N^\varepsilon]\!]$ we have

$$\max(\langle \lambda_i - \lambda_{i_1}, \lambda_j\rangle_\mu, \langle \lambda_i, \lambda_j - \lambda_{j_1}\rangle_\mu) \ll \langle \lambda_i, \lambda_j\rangle_\mu.$$

This is expected to be true since $\langle \lambda_a; \lambda_b\rangle \sim (a/b)^{1/3}$ for $a \ll b$ should imply that $\langle \lambda_a - \lambda_{a'}; \lambda_b\rangle \sim N^\varepsilon \partial_a (a/b)^{1/3}$ if $b - a \leqslant N^\varepsilon$, therefore

$$\max(N^\varepsilon \partial_i (i/j)^{1/3}, N^\varepsilon \partial_j (i/j)^{1/3}) \sim N^\varepsilon \max(i^{-2/3} j^{-1/3}, i^{1/3} j^{-4/3}) \ll (i/j)^{1/3}.$$

(ii) For $\beta = 2$, the spectral measure is a determinantal point process (with kernel K_N normalized such that $N^{-1}K_N(x,x) \to (2\pi)^{-1}\sqrt{(4-x^2)_+}$), and an elementary calculation gives

$$\langle \mathcal{N}(\gamma_i), \mathcal{N}(\gamma_j))\rangle_\mu = \iint_{(-\infty,\gamma_i]\times[\gamma_j,\infty)} |K_N(x,y)|^2 \mathrm{d}x\mathrm{d}y. \tag{E.2}$$

(iii) Via the Christoffel-Darboux formula, the correlation kernel can be expressed in terms of two successive Hermite polynomials. The Plancherel-Rotach asymptotics then allow us to prove that in the above integral, the main contribution comes from the domain $I \times J = [-2 + N^{-2/3+\varepsilon}, \gamma_i] \times [\gamma_j, 0]$ where $\varepsilon > 0$ is small enough. In this domain one can prove (see (5.4) in [34]) that $K_N(x, y)$ is asymptotically equivalent to

$$\frac{\mathrm{Ai}(-(nF(x))^{2/3})\mathrm{Ai}'(-(nF(y))^{2/3}) - \mathrm{Ai}'(-(nF(x))^{2/3})\mathrm{Ai}(-(nF(y))^{2/3})}{x - y},$$

where Ai is the Airy function and $F(x) \sim (x + 2)^{3/2}$ as x decreases to -2. Thanks to the estimates

$$\mathrm{Ai}(-r) \sim r^{-1/4} \left(\cos(2/3\, r^{3/2} - \frac{\pi}{4}) + \mathrm{O}(r^{-3/2})\right),$$
$$\mathrm{Ai}'(-r) \sim r^{1/4} \left(\sin(2/3\, r^{3/2} - \frac{\pi}{4}) + \mathrm{O}(r^{-3/2})\right),$$

as $r \to \infty$, we can approximate $K_N(x, y)$ by

$$\left(\frac{2 + y}{2 + x}\right)^{1/4} \frac{\cos(\frac{2}{3} n^{3/2}(2 + x) - \frac{\pi}{4}) \sin(\frac{2}{3} n^{3/2}(2 + y) - \frac{\pi}{4})}{x - y}, \tag{E.3}$$

by noting that in $I \times J$ we have $2 + y \gg 2 + x$.

(iv) The end of these heuristics consists in the following calculation, where we use that the square of the oscillating term in (E.3) averages to 1/2 (note that the frequencies go to ∞), and we note $U \times V = [N^\varepsilon, i^{2/3}] \times [j^{2/3}, N^{2/3}]$:

$$\iint_{I \times J} |K_N(x, y)|^2 \mathrm{d}x\mathrm{d}y \sim \iint_{U \times V} \left(\frac{v}{u}\right)^{1/2} \frac{1}{(u - v)^2} \mathrm{d}u\mathrm{d}v \sim \int_U u^{-1/2}\mathrm{d}u \int_V v^{-3/2}\mathrm{d}v \sim \left(\frac{i}{j}\right)^{1/3}.$$

One concludes using the above equation, (E.1) and (E.2).

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