Gaussian fluctuations of the determinant of Wigner Matrices

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We prove that the logarithm of the determinant of a Wigner matrix satisfies a central limit theorem in the limit of large dimension. Previous results about fluctuations of such determinants required that the first four moments of the matrix entries match those of a Gaussian [54]. Our work treats symmetric and Hermitian matrices with centered entries having the same variance and subgaussian tail. In particular, it applies to symmetric Bernoulli matrices and answers an open problem raised in [55]. The method relies on (1) the observable introduced in [10] and the stochastic advection equation it satisfies, (2) strong estimates on the Green function as in [12], (3) fixed energy universality [8], (4) a moment matching argument [53] using Green's function comparison [21].

1 Introduction

In this paper, we address the universality of the determinant of a class of random Hermitian matrices. Before discussing results specific to this symmetry assumption, we give a brief history of results in the non-Hermitian setting. In both settings, a priori bounds preceded estimates on moments of determinants, and the distribution of determinants for integrable models of random matrices. The universality of such determinants has then been the subject of recent active research.

1.1 Non-Hermitian matrices. Early papers on this topic treat non-Hermitian matrices with independent and identically distributed entries. More specifically, Szekeres and Turán first studied an extremal problem on the determinant of ± 1 matrices [50]. In the 1950s, a series of papers [23, 24, 44, 47, 56] calculated moments of the determinant of random matrices of fixed size (see also [28]). In general, explicit formulae are unavailable for high order moments of the determinant except when the entries of the matrix have particular distribution (see, for example, [17] and the references therein). Estimates for the moments and the Chebyshev inequality give upper bounds on the magnitude of the determinant.

Along a different line of research, for an $N \times N$ non-Hermitian random matrix A, Erdős asked whether $\det A$ is non-zero with probability tending to one as N tends to infinity. In [33, 35], Kolmós proved that for random matrices with Bernoulli entries, indeed $\det A \neq 0$ with probability converging to 1 with N. In fact, this method works for more general models, and following [33], [11, 32, 51, 52] give improved, exponentially

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small bounds on the probability that $\det A = 0$.

In [51], the authors made the first steps towards quantifying the typical size of $|\det A|$, proving that for Bernoulli random matrices, with probability tending to 1 as N tends to infinity,

$$\sqrt{N!}\exp\left(-c\sqrt{N\log N}\right) \leqslant |\det A| \leqslant \omega(N)\sqrt{N!}, \tag{1.1}$$

for any function $\omega(N)$ tending to infinity with N. In particular, with overwhelming probability

$$\log|\det A| = \left(\frac{1}{2} + \mathrm{o}(1)\right) N\log N.$$

In [30], Goodman considered A with independent standard real Gaussian entries. In this case, he was able to express $|\det A|^2$ as the product of independent chi-square variables. This enables one to identify the asymptotic distribution of $\log|\det A|$. Indeed, one can prove that

$$\frac{\log|\det A| - \frac{1}{2}\log N! + \frac{1}{2}\log N}{\sqrt{\frac{1}{2}\log N}} \to \mathscr{N}(0,1), \tag{1.2}$$

(see [48]). In the case of A with independent complex Gaussian entries, a similar analysis yields

$$\frac{\log|\det A| - \frac{1}{2}\log N! + \frac{1}{4}\log N}{\sqrt{\frac{1}{4}\log N}} \to \mathscr{N}(0,1).$$

In [42], the authors proved (1.2) holds under just an exponential decay hypothesis on the entries. Their method yields an explicit rate of convergence and extends to handle the complex case. Then in [5], the authors extended (1.2) to the case where the matrix entries only require bounded fourth moment.

The analysis of determinants of non-Hermitian random matrices relies crucially on the assumption that the rows of the random matrix are independent. The fact that this independence no longer holds for Hermitian random matrices forces one to look for new methods to prove similar results to those of the non-Hermitian case. Nevertheless, the history of this problem mirrors the history of the non-Hermitian case.

1.2 Hermitian matrices. In the 1980s, Weiss posed the Hermitian analogs of [33, 35] as an open problem. This problem was solved, many years later in [15], and then in [53, Theorem 34] the authors proved the Hermitian analog of (1.1). This left open the question of describing the limiting distribution of the determinant.

In [16], Delannay and Le Caër used the explicit formula for the joint distribution of the eigenvalues to prove that for H an $N \times N$ matrix drawn from the GUE,

$$\frac{\log|\det H| - \frac{1}{2}\log N! + \frac{1}{4}\log N}{\sqrt{\frac{1}{2}\log N}} \to \mathscr{N}(0,1). \tag{1.3}$$

Analogously, one has

$$\frac{\log|\det H| - \frac{1}{2}\log N! + \frac{1}{4}\log N}{\sqrt{\log N}} \to \mathscr{N}(0,1) \tag{1.4}$$

when H is drawn from the GOE. Proofs of these central limit theorems also appear in [7, 13, 18, 54]. For related results concerning other models of random matrices, see [49] and the references therein.

While the authors of [54] give their own proof of (1.3) and (1.4), their main interest is to establish such a result in the more general setting of Wigner matrices. Indeed, they show that in (1.4), we may replace H by W, a Wigner matrix whose entries' first four moments match those of $\mathscr{N}(0,1)$. They also prove the analogous result in the complex case. In this paper, we will relax this four moment matching assumption to a two moment matching assumption (see Theorem 1.2).

Finally, we mention that new interest in averages of determinants of random (Hermitian) matrices has emerged from the study of complexity of high-dimensional landscapes [4, 27].

1.3 Statement of results: The determinant. This subsection gives our main result and suggests extensions in connection with the general class of log-correlated random fields. Our theorems apply to Wigner matrices as defined below.

Definition 1.1. *A complex Wigner matrix, $W = (w_{ij})$, is an $N \times N$ Hermitian matrix with entries*

$$W_{ii} = \sqrt{\frac{1}{N}}\, x_{ii},\ i = 1, \ldots, N, \quad W_{ij} = \frac{1}{\sqrt{2N}}\left(x_{ij} + \mathrm{i} y_{ij}\right),\ 1 \leqslant i < j \leqslant N.$$

Here $\{x_{ii}\}_{1\leqslant i\leqslant N}$, $\{x_{ij}\}_{1\leqslant i<j\leqslant N}$, $\{y_{ij}\}_{1\leqslant i<j\leqslant N}$ are independent identically distributed random variables satisfying $\mathbb{E}\left(x_{ij}\right) = 0, \mathbb{E}\left(x_{ij}^2\right) = \mathbb{E}\left(y_{ij}^2\right) = 1$. We assume further that the common distribution ν of $\{x_{ii}\}_{1\leqslant i\leqslant N}$, $\{x_{ij}\}_{1\leqslant i<j\leqslant N}$, $\{y_{ij}\}_{1\leqslant i<j\leqslant N}$, has subgaussian decay, i.e. there exists $\delta_0 > 0$ such that

$$\int_{\mathbb{R}} e^{\delta_0 x^2} \mathrm{d}\nu(x) < \infty. \tag{1.5}$$

In particular, this means that all the moments of the entries of the matrix are bounded. In the special case $\nu = \mathscr{N}(0,1)$, W is said to be drawn from the Gaussian Unitary Ensemble (GUE).

Similarly, we define a real Wigner matrix to have entries of the form $W_{ii} = \sqrt{\frac{2}{N}} x_{ii}$, $W_{ij} = \sqrt{\frac{1}{N}} x_{ij}$, where $\{x_{ij}\}_{1\leqslant i,j\leqslant N}$ are independent identically distributed random variables satisfying $\mathbb{E}\left(x_{ij}\right) = 0, \mathbb{E}\left(x_{ij}^2\right) = 1$. As in the complex case, we assume the common distribution ν satisfies (1.5). In the special case $\nu = \mathscr{N}(0,1)$, W is said to be drawn from the Gaussian Orthogonal Ensemble (GOE).

Our main result extends (1.3) and (1.4) to the above class of Wigner matrices. In particular, this answers a conjecture from [55, Section 8], which asserts that the central limit theorem (1.4) holds for Bernoulli (± 1) matrices. Note that in the following statement, our centering differs from (1.3) and (1.4) because we normalize our matrix entries to have variance of size N^{-1}.

Theorem 1.2. *Let W be a real Wigner matrix satisfying (1.5). Then*

$$\frac{\log|\det W| + \frac{N}{2}}{\sqrt{\log N}} \to \mathscr{N}(0,1). \tag{1.6}$$

If W is a complex Wigner matrix satisfying (1.5), then

$$\frac{\log|\det W| + \frac{N}{2}}{\sqrt{\frac{1}{2}\log N}} \to \mathscr{N}(0,1). \tag{1.7}$$

Assumption (1.5) may probably be relaxed to a finite moment assumption, but we will not pursue this direction here. Similarly, it is likely that the matrix entries do not need to be identically distributed; only the first two moments need to match. However we consider the case of a unique ν in this paper.

Remark 1.3. *Let H be drawn from the GUE normalized so that in the limit as $N \to \infty$, the distribution of its eigenvalues is supported on $[-1,1]$, and let*

$$D_N(x) = -\log\left|\det\left(H - x\right)\right|.$$

In [36], Krasovsky proved that for $x_k \in (-1,1)$, $k = 1, \ldots, m$, $x_j \neq x_k$, uniformly in $\Re\left(\alpha_k\right) > -\frac{1}{2}$, $\mathbb{E}\left(e^{-\sum_{k=1}^m \alpha_k D_N(x_k)}\right)$ is asymptotic to

$$\prod_{k=1}^m \left(C\left(\frac{\alpha_k}{2}\right)\left(1 - x_k^2\right)^{\frac{\alpha_k^2}{8}} N^{\frac{\alpha_k^2}{4}} e^{\frac{\alpha_k N}{2}\left(2x_k^2 - 1 - 2\log 2\right)}\right) \prod_{1\leqslant\nu<\mu\leqslant m} \left(2\left|x_\nu - x_\mu\right|\right)^{-\frac{\alpha_\nu\alpha_\mu}{2}} \left(1 + \mathrm{O}\left(\frac{\log N}{N}\right)\right), \tag{1.8}$$

as $N \to \infty$. Here $C(\cdot)$ is the Barnes function. Since the above estimate holds uniformly for $\Re\left(\alpha_k\right) > -\frac{1}{2}$, (1.8) shows that letting

$$\widetilde{D}_N(x) = \frac{D_N(x) + N\left(x^2 - \frac{1}{2} - \log 2\right)}{\sqrt{\frac{1}{2}\log N}},$$

the vector $\left(\widetilde{D}_N(x_1), \dots, \widetilde{D}_N(x_m)\right)$ *converges in distribution to a collection of* m *independent standard Gaussians. Our proof of Theorem 1.2 automatically extends this result to Hermitian Wigner matrices as defined above. If one were to prove an analogous convergence for the GOE, our proof of Theorem 1.2 would extend the result to real symmetric Wigner matrices as well.*

Remark 1.4. *We note that (1.8) was proved for fixed, distinct* x_k*'s. If (1.8) holds for collapsing* x_k*'s, this means that fluctuations of the log-characteristic polynomial of the GUE become log-correlated for large dimension, as in the case of the Circular Unitary Ensemble [9]. More specifically, let* $\widetilde{D}_N(\cdot)$ *be as above, and let* Δ *denote the distance between two points* x, y *in* $(-1,1)$*. For* $\Delta \geqslant 1/N$*, we expect the covariance between* $\widetilde{D}_N(x)$ *and* $\widetilde{D}_N(y)$ *to behave like* $\frac{\log(1/\Delta)}{\log N}$*, and for* $\Delta \leqslant 1/N$*, we expect it to converge to 1.*

Our method automatically establishes the content of Remark 1.4 for Wigner matrices, conditional on the knowledge of GOE and GUE cases. The exact statement is as follows, and we omit the proof, strictly similar to Theorem 1.2. Denote

$$L_N(z) = \log|\det(W - z)| - N \int_{-2}^{2} \log|x - z| \, \mathrm{d}\rho_{\mathrm{sc}}(x).$$

Theorem 1.5. *Let* W *be a real Wigner matrix satisfying (1.5). Let* $\ell \geqslant 1$*,* $\kappa > 0$ *and let* $(E_N^{(1)})_{N\geqslant 1}, \dots, (E_N^{(\ell)})_{N\geqslant 1}$ *be energy levels in* $[-2+\kappa, 2-\kappa]$*. Assume that for all* $i \neq j$*, for some constants* c_{ij} *we have*

$$\frac{\log|E_N^{(i)} - E_N^{(j)}|}{-\log N} \to c_{ij} \in [0, \infty]$$

as $N \to \infty$*. Then*

$$\frac{1}{\sqrt{\frac{1}{2}\log N}} \left(L_N\left(\left(E_N^{(1)}\right)\right), \dots, L_N\left(\left(E_N^{(\ell)}\right)\right) \right) \tag{1.9}$$

converges in distribution to a Gaussian vector with covariance $(\min(1, c_{ij}))_{1\leqslant i,j\leqslant N}$ *(with diagonal* 1 *by convention), provided the same result holds for GOE.*

The same result holds for Hermitian Wigner matrices, assuming it is true in the GUE case, up to a change in the normalization from $\sqrt{\frac{1}{2}\log N}$ *to* $\sqrt{\log N}$ *in (1.9).*

Theorem 1.5 says L_N converges to a log-correlated field, provided this result holds for the Gaussian ensembles. It therefore suggests that the universal limiting behavior of extrema and convergence to Gaussian multiplicative chaos conjectured for unitary matrices in [25] extends to the class of Wigner matrices. Towards these conjectures, [3, 14, 26, 37, 46] proved asymptotics on the maximum of characteristic polynomials of circular unitary and invariant ensembles, and [6, 43, 57] established convergence to the Gaussian multiplicative chaos, for the same models. We refer to [2] for a survey on log-correlated fields and their connections with random matrices, branching processes, the Gaussian free field, and analytic number theory.

1.4 Statement of results: Fluctuations of Individual Eigenvalues. With minor modifications, the proof of Theorem 1.2 also extends the results of [31] and [45] which describe the fluctuations of individual eigenvalues in the GUE and GOE cases, respectively. By adapting the method of [53], [45] proves the following theorem under the assumption that the first four moments of the matrix entries match those of a standard Gaussian. In Appendix B, we show that the individual eigenvalue fluctuations of the GOE (GUE) also hold for real (complex) Wigner matrices in the sense of Definition 1.1. In particular, the fluctuations of eigenvalues of Bernoulli matrices are Gaussian in the large dimension limit, which answers a question from [55].

To state the following theorem, we follow the notation of Gustavsson [31] and write $k(N) \sim N^{\theta}$ to mean that $k(N) = h(N)N^{\theta}$ where h is a function such that for all $\varepsilon > 0$, for large enough N,

$$N^{-\varepsilon} \leqslant h(N) \leqslant N^{\varepsilon}. \tag{1.10}$$

In the following, γ_k denotes the k^{th} quantile of the semicircle law,

$$\frac{1}{2\pi} \int_{-2}^{\gamma_k} \sqrt{(4-x^2)_+} \mathrm{d}x = \frac{k}{N}. \tag{1.11}$$

Theorem 1.6. *Let W be a Wigner matrix satisfying (1.5) with eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_N$. Consider $\{\lambda_{k_i}\}_{i=1}^m$ such that $0 < k_{i+1} - k_i \sim N^{\theta_i}, 0 < \theta_i \leqslant 1$, and $k_i/N \to a_i \in (0,1)$ as $N \to \infty$. Let*

$$X_i = \frac{\lambda_{k_i} - \gamma_{k_i}}{\sqrt{\frac{4 \log N}{\beta\left(4-\gamma_{k_i}^2\right)N^2}}}, \quad i = 1, \ldots, m, \tag{1.12}$$

with $\beta = 1$ for real Wigner matrices, and $\beta = 2$ for complex Wigner matrices. Then as $N \to \infty$,

$$\mathbb{P}\left\{X_1 \leqslant \xi_1, \ldots, X_m \leqslant \xi_m\right\} \to \Phi_\Lambda\left(\xi_1, \ldots, \xi_m\right),$$

where Φ_Λ is the cumulative distribution function for the m-dimensional normal distribution with covariance matrix $\Lambda_{i,j} = 1 - \max\{\theta_k : i \leqslant k < j < m\}$ if $i < j$, and $\Lambda_{i,i} = 1$.

The above theorem has been known to follow from the homogenization result in [8] (this technique gives a simple expression for the relative *individual* positions of coupled eigenvalues from GOE and Wigner matrices) and fluctuations of mesoscopic linear statistics; see [38] for a proof of eigenvalue fluctuations for Wigner and invariant ensembles. However, the technique from [8] is not enough for Theorem 1.2, as the determinant depends on the positions of *all* eigenvalues.

1.5 Outline of the proof. In this section, we give the main steps of the proof of Theorem 1.2. Our outline discusses the real case, but the complex case follows the same scheme.

The main conceptual idea of the proof follows the three step strategy of [19, 20]. With a priori localization of eigenvalues (step one, [12, 22]), one can prove that the determinant has universal fluctuations after a adding a small Gaussian noise (this second step relies on a stochastic advection equation from [10]). The third step proves by a density argument that the Gaussian noise does not change the distribution of the determinant, thanks to a perturbative moment matching argument as in [21, 53]. We include Figure 1 below to help summarize the argument.

First step: small regularization. In Section 2, with Theorems 2.2 and 2.4, we reduce the proof of Theorem 1.2 to showing the convergence

$$\frac{\log|\det(W + \mathrm{i}\eta_0)| + c_N}{\sqrt{\log N}} \to \mathscr{N}(0,1) \tag{1.13}$$

with some explicit deterministic c_N, and the small regularization parameter

$$\eta_0 = \frac{e^{(\log N)^{\frac{1}{4}}}}{N}. \tag{1.14}$$

Second step: universality after coupling. Let M be a symmetric matrix which serves as the initial condition for the matrix Dyson's Brownian Motion (DBM) given by

$$\mathrm{d}M_t = \frac{1}{\sqrt{N}}\mathrm{d}B^{(t)} - \frac{1}{2}M_t \mathrm{d}t. \tag{1.15}$$

Here $B^{(t)}$ is a symmetric $N \times N$ matrix such that $B_{ij}^{(t)}$ $(i < j)$ and $B_{ii}^{(t)}/\sqrt{2}$ are independent standard Brownian motions. The above matrix DBM induces a collection of independent standard Brownian motions (see [1]), $\tilde{B}_t^{(k)}/\sqrt{2}$, $k = 1, \ldots, N$ such that the eigenvalues of M satisfy the system of stochastic differential equations

$$\mathrm{d}x_k(t) = \frac{\mathrm{d}\tilde{B}_t^{(k)}}{\sqrt{N}} + \left(\frac{1}{N}\sum_{l \neq k} \frac{1}{x_k(t) - x_l(t)} - \frac{1}{2}x_k(t)\right)\mathrm{d}t \tag{1.16}$$

with initial condition given by the eigenvalues of M. It has been known since [41] that the system (1.16) has a unique strong solution. With this in mind, we follow [8] and introduce the following coupling scheme. First, run the matrix DBM taking $\tilde{W}_0$, a Wigner matrix, as the initial condition. Using the induced Brownian motions, run the dynamics given by (1.16) using the eigenvalues $y_1 < y_2 < \cdots < y_N$ of $\tilde{W}_0$ as the initial condition. Call the solution to this system $\boldsymbol{y}(\tau)$. Using the very same (induced) Brownian motions, run the dynamics given by (1.16) again, this time using the eigenvalues of a GOE matrix, $\boldsymbol{x}(0)$, as the initial condition. Call the solution to this system $\boldsymbol{x}(\tau)$.

Now fix $\varepsilon > 0$ and let

$$\tau = N^{-\varepsilon}. \tag{1.17}$$

Using Lemma 3.1, we show that

$$\frac{\sum_{k=1}^N \log|x_k(\tau)+\mathrm{i}\eta_0| - \sum_{k=1}^N \log|y_k(\tau)+\mathrm{i}\eta_0|}{\sqrt{\log N}} \tag{1.18}$$

and

$$\frac{\sum_{k=1}^N \log|x_k(0)+z_\tau| - \sum_{k=1}^N \log|y_k(0)+z_\tau|}{\sqrt{\log N}} \tag{1.19}$$

are very close. Here z_τ is as in (3.5) with $z = \mathrm{i}\eta_0$. The significance of this is that since $z_\tau \sim \mathrm{i}\tau$, we can use Lemma A.1 and well-known central limit theorems which apply to nearly macroscopic scales to show that (1.19) has variance of order ε. Consequently, (1.18) is also small, and since $\boldsymbol{x}(\tau)$ is distributed as the eigenvalues of a GOE matrix, we have proved universality of the regularized determinant after coupling.

Third step: moment matching. In Section 4, we conclude the proof of Theorem 1.2. First, we choose $\tilde{W}_0$ so that $\tilde{W}_\tau$ and W have entries whose first four moments are close, as in [21]. With this approximate moment matching, we use a perturbative argument, as in [54], to prove that (1.13) holds for W if and only if it holds for $\tilde{W}_\tau$. But as (1.18) is small, this means (1.13) holds for W if and only if it holds for a GOE matrix. By (1.4), this concludes the proof.

$$\begin{tikzcd}
 & W \\
\tilde{W}_0 \arrow[r, "\text{Matrix DBM } \mathrm{d}B_{ij}"] & \tilde{W}_\tau \arrow[u, leftrightarrow, "\text{Moment Matching (3)}"'] \\
\boldsymbol{y}(0) \arrow[r, "\text{Eigenvalues DBM } \mathrm{d}\tilde{B}_k"'] & \boldsymbol{y}(\tau) \\
\boldsymbol{x}(0) \arrow[r, "\text{Eigenvalues DBM } \mathrm{d}\tilde{B}_k"] & \boldsymbol{x}(\tau) \arrow[u, "\text{Coupling (2)}"']
\end{tikzcd}$$

Figure 1: We show (1.6) holds for $\tilde{W}_\tau$ if and only if it holds for W, and we prove that (1.6) holds for $\boldsymbol{x}(\tau)$ if and only if (1.6) holds for $\tilde{W}_\tau$. Since $\boldsymbol{x}(\tau)$ is distributed as the eigenvalues of a GOE matrix, it satisfies (1.4) and we conclude the proof. Note that $\log|\det \tilde{W}_\tau| = \sum \log|y_k(\tau)|$ pathwise because B induces $\tilde{B}$.

1.6 Notation. We shall make frequent use of the notations s_W and m_{sc} in the remainder of this paper. We state their definitions here for easy reference. Let W be a Wigner matrix with eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_N$. For $\Im(z) > 0$, define

$$s_W(z) = \frac{1}{N}\sum_{k=1}^N \frac{1}{\lambda_k - z}, \tag{1.20}$$

the Stieltjes transform of W. Next, let

$$m_{sc}(z) = \frac{-z+\sqrt{z^2-4}}{2}, \tag{1.21}$$

where the square root $\sqrt{z^2-4}$ is chosen with the branch cut in $[-2,2]$ so that $\sqrt{z^2-4} \sim z$ as $z \to \infty$. Note that

$$m_{sc}(z) + \frac{1}{m_{sc}(z)} + z = 0. \tag{1.22}$$

Finally, throughout this paper, unless indicated otherwise, C (c) denotes a large (small) constant independent of all other parameters of the problem. It may vary from line to line.

2 Initial Regularization

Let $y_1 < y_2 < \cdots < y_N$ denote the eigenvalues of W, a real Wigner matrix satisfying (1.5). We first prove we only need to show Theorem 1.2 for a slight regularization of the logarithm.

Proposition 2.1. *Set*

$$g(\eta) = \sum_k (\log|y_k + \mathrm{i}\eta| - \log|y_k|) - \int_0^\eta N\Im(m_{sc}(\mathrm{i}s))\,\mathrm{d}s$$

and recall $\eta_0 = \frac{e^{(\log N)^{\frac{1}{4}}}}{N}$ *as in (1.14). Then we have the convergence in probability*

$$\frac{g(\eta_0)}{\sqrt{\log N}} \to 0.$$

To prove Proposition 2.1, we will use Theorems 2.2 and 2.4 as input. In [12], Theorem 2.2 is stated for complex Wigner matrices, however, the argument there proves the same statement for real Wigner matrices.

Theorem 2.2 (Theorem 1 in [12])**.** *Let W be a Wigner matrix and fix $\tilde{\eta} > 0$. For any $\tilde{E} > 0$, there exist constants $M_0, N_0, C, c, c_0 > 0$ such that*

$$\mathbb{P}\left(|\Im(s_W(E+\mathrm{i}\eta)) - \Im(m_{sc}(E+\mathrm{i}\eta))| \geqslant \frac{K}{N\eta}\right) \leqslant \frac{(Cq)^{cq^2}}{K^q}$$

for all $\eta \leqslant \tilde{\eta}$, $|E| \leqslant \tilde{E}$, $K > 0$, $N > N_0$ such that $N\eta > M_0$, and $q \in \mathbb{N}$ with $q \leqslant c_0 (N\eta)^{\frac{1}{8}}$.

Remark 2.3. *In [22], the authors proved that for some positive constant C_0, and N large enough,*

$$|s_W(E+\mathrm{i}\eta) - m_{sc}(E+\mathrm{i}\eta)| \leqslant \frac{e^{C_0(\log\log N)^2}}{N\eta}$$

holds with high probability. Though this estimate is weaker than the estimate of Theorem 2.2, it holds for a more general model of Wigner matrix in which the entries of the matrix need not have identical variances. On the other hand, we require the stronger estimate in Theorem 2.2 in our proof of Proposition 2.1, and so we restrict ourselves to Wigner matrices as defined in Definition 1.1. The proof of Lemma A.1 also relies on Definition 1.1.

Theorem 2.4 (Theorem 2.2 in [8])**.** *Let ρ_1 denote the first correlation function for the eigenvalues of an $N \times N$ Wigner matrix, and let $\rho(x) = \frac{1}{2\pi}\sqrt{(4-v^2)_+}$. Then for any $F : \mathbb{R} \to \mathbb{R}$ continuous and compactly supported, and for any $\kappa > 0$, we have,*

$$\lim_{N\to\infty} \sup_{E\in[-2+\kappa,2-\kappa]} \left| \frac{1}{\rho(E)} \int F(v)\rho_1\left(E + \frac{v}{N\rho(E)}\right)\mathrm{d}v - \int F(v)\rho(v)\,\mathrm{d}v \right| = 0. \tag{2.1}$$

Remark 2.5. *In fact Theorem 2.2 in [8] makes a much stronger statement, namely it states the analogous convergence for all correlation functions in the case of generalized Wigner matrices.*

Corollary 2.6. *For any small fixed $\kappa, \gamma > 0$ there exists $C, N_0 > 0$ such that for any $N \geqslant N_0$ and any interval $I \subset [-2+\kappa, 2-\kappa]$ we have*

$$\mathbb{E}(|\{y_k : y_k \in I\}|) \leqslant CN|I| + \gamma.$$

Proof. In Theorem 2.4, choosing F to be an indicator of an interval of length 1 gives an expected value O(1). Since the statement of Theorem 2.4 holds uniformly in E, we may divide the interval I into sub-intervals of length order $1/N$ to conclude. □

Corollary 2.7. *Let $E \in [-2+\kappa, 2-\kappa]$ be fixed and $I_\beta = (E-\beta/2, E+\beta/2)$ with $\beta = \mathrm{o}(N^{-1})$. Then*

$$\lim_{N\to\infty} \mathbb{P}\left(|\{y_k \in I_\beta\}| = 0\right) = 1.$$

Proof. Let ε be any fixed small constant. Let f be fixed, smooth, positive, equal to 1 on $[-1,1]$ and 0 on $[-2,2]^c$. Then

$$\mathbb{P}\left(|\{y_k \in I_\beta\}| \geqslant 1\right) \leqslant \mathbb{E}\left(|\{y_k \in I_\beta\}|\right) \leqslant \mathbb{E}\left(\sum_k f\left(N(y_k - E)/\varepsilon\right)\right) \leqslant 10\varepsilon,$$

where the last bound holds for large enough N by Theorem 2.4. □

Proof of Proposition 2.1. We first choose $\tilde\eta < \eta_0$ so that we can use Theorem 2.2 to estimate

$$\mathbb{E}\left(|g(\eta_0) - g(\tilde\eta)|\right),$$

and then take care of the remaining error using Corollaries 2.6 and 2.7. Let

$$\tilde\eta = \frac{d_N}{N}, \quad \text{with} \quad d_N = (\log N)^{\frac14},$$

and observe that

$$\mathbb{E}\left(|g(\eta_0) - g(\tilde\eta)|\right) = \mathbb{E}\left(\left|\int_{\tilde\eta}^{\eta_0} N\Im\left(s_W(\mathrm{i}t) - m_{sc}(\mathrm{i}t)\right)\mathrm{d}t\right|\right) \leqslant \int_{\tilde\eta}^{\eta_0} \mathbb{E}(N\left|\Im\left(s_{W_1}(\mathrm{i}t) - m_{sc}(\mathrm{i}t)\right|\right))\,\mathrm{d}t. \tag{2.2}$$

In estimating the right hand side above, we will use the notation

$$\Delta(t) = \left|\Im\left(s_{W_1}(\mathrm{i}t) - m_{sc}(\mathrm{i}t)\right|\right..$$

For N sufficiently large, by Theorem 2.2 with $q=2$, we can write the right hand side of (2.2) as

$$\begin{aligned}\int_{\tilde\eta}^{\eta_0}\int_0^\infty \mathbb{P}\left(N\Delta(t) > u\right)\mathrm{d}u\mathrm{d}t &= \int_{\tilde\eta}^{\eta_0}\left(\int_0^1 \mathbb{P}\left(\Delta(t) > \frac{K}{Nt}\right)\frac{\mathrm{d}K}{t} + \int_1^\infty \mathbb{P}\left(\Delta(t) > \frac{K}{Nt}\right)\frac{\mathrm{d}K}{t}\right)\mathrm{d}t\\ &\leqslant \int_{\tilde\eta}^{\eta_0}\left(\frac1t + \int_1^\infty \frac{C}{K^2}\frac{dK}{t}\right)\mathrm{d}t \leqslant (1+C)\log\left(\frac{\eta_0}{\tilde\eta}\right) = \mathrm{o}\left(\sqrt{\log N}\right).\end{aligned} \tag{2.3}$$

Next we estimate $\sum_k\left(\log|y_k + \mathrm{i}\tilde\eta| - \log|y_k|\right)$, and this will give us a bound for $\mathbb{E}\left(|g(\tilde\eta)|\right)$. Taylor expansion yields

$$\sum_{|y_k|>\tilde\eta}\left(\log|y_k + \mathrm{i}\tilde\eta| - \log|y_k|\right) \leqslant \sum_{|y_k|>\tilde\eta}\frac{\tilde\eta^2}{y_k^2}.$$

Define $N_1(u) = |\{y_k : \tilde\eta \leqslant |y_k| \leqslant u\}|$. Using integration by parts and Corollary 2.6, we have

$$\mathbb{E}\left(\sum_{|y_k|>\tilde\eta}\frac{\tilde\eta^2}{y_k^2}\right) = \mathbb{E}\left(\int_{\tilde\eta}^\infty \frac{\tilde\eta^2}{y^2}\,\mathrm{d}N_1(y)\right) = 2\tilde\eta^2\int_{\tilde\eta}^\infty \frac{\mathbb{E}\left(N_1(y)\right)}{y^3}\,\mathrm{d}y = \mathrm{O}\left(d_N\right). \tag{2.4}$$

We now estimate $\sum_{|y_k|\leqslant\tilde\eta}\left(\log|y_k + \mathrm{i}\tilde\eta| - \log|y_k|\right)$. We consider two cases. First, let $A_N = b_N/N$ for some very small b_N, for example

$$b_N = e^{-(\log N)^{\frac14}}.$$

For $u > 0$ we denote $N_2(u) = |\{y_k : A_N < |y_k| \leqslant u\}|$. Then again using integration by parts and Corollary 2.6 we obtain

$$\begin{aligned}\mathbb{E}\left(\sum_{A_N<|y_k|<\tilde\eta}\left(\log|y_k + \mathrm{i}\tilde\eta| - \log|y_k|\right)\right) &= \mathbb{E}\left(\int_{A_N}^{\tilde\eta}\left(\log|y + \mathrm{i}\tilde\eta| - \log|y|\right)\mathrm{d}N_2(y)\right)\\ &\leqslant \log\left(\sqrt2\right)\mathbb{E}\left(N_2(\tilde\eta)\right) + \int_{A_N}^{\tilde\eta}\frac{\mathbb{E}\left(N_2(y)\right)}{y}\mathrm{d}y = \mathrm{O}\left(d_N + d_N\log\left(\frac{d_N}{b_N}\right)\right) = \mathrm{o}\left(\sqrt{\log N}\right).\end{aligned}$$

It remains to estimate $\sum_{|y_k|<A_N}(\log|y_k+\mathrm{i}\tilde\eta|-\log|y_k|)$. By Corollary 2.7, we have

$$\mathbb{P}\left(\sum_{|y_k|<A_N}(\log|y_k+\mathrm{i}\tilde\eta|-\log|y_k|)=0\right)\geqslant \mathbb{P}\left(|\{y_k\in[-A_N,A_N]\}|=0\right)\to 1. \tag{2.5}$$

The estimates (2.3) and (2.4) along with Markov's inequality, and the bound (2.5), conclude the proof. □

3 Coupling of Determinants

In this section, we use the coupled Dyson Brownian Motion introduced in [8] to compare (1.18) and (1.19). Define $\tilde W_\tau$ by running the matrix Dyson Brownian Motion (1.15) with initial condition $\tilde W_0$ where $\tilde W_0$ is a Wigner matrix with eigenvalues $\boldsymbol{y}$. Recall that this induces a collection of Brownian motions $\tilde B_t^{(k)}$ so that the system (1.16) with initial condition $\boldsymbol{y}$ has a (unique strong) solution $\boldsymbol{y}(\cdot)$, and $\boldsymbol{y}(\tau)$ are the eigenvalues of $\tilde W_\tau$. Using the same (induced) Brownian motions as we used to define $\boldsymbol{y}(\tau)$, define $\boldsymbol{x}(\tau)$ by running the dynamics (1.16) with initial condition given by the eigenvalues of a GOE matrix. We now prove Proposition 3.2 which says that (1.18) and (1.19) are asymptotically equal in law, with main tool being the following Lemma 3.1.

To study the coupled dynamics of $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$, we follow [10,39]. For $\nu\in[0,1]$, let

$$\lambda_k^\nu(0)=\nu x_k+(1-\nu)\,y_k \tag{3.1}$$

where $\boldsymbol{x}$ is the spectrum of a GOE matrix, and $\boldsymbol{y}$ is the spectrum of $\tilde W_0$. With this initial condition, we denote the (unique strong) solution to (1.16) by $\boldsymbol{\lambda}^{(\nu)}(t)$. Note that $\boldsymbol{\lambda}^{(0)}(\tau)=\boldsymbol{y}(\tau)$ and $\boldsymbol{\lambda}^{(1)}(\tau)=\boldsymbol{x}(\tau)$. Let

$$f_t^{(\nu)}(z)=e^{-\frac{t}{2}}\sum_{k=1}^N\frac{u_k(t)}{\lambda_k^{(\nu)}(t)-z},\quad u_k(t)=\frac{\mathrm{d}}{\mathrm{d}\nu}\lambda_k^{(\nu)}(t), \tag{3.2}$$

(see [39] for existence of this derivative). A key observation of [10] is that the time evolution of $f_t^{(\nu)}$ is, at leading order,

$$\partial_t f_t^{(\nu)}\approx\frac{\sqrt{z^2-4}}{2}\partial_z f_t^{(\nu)}, \tag{3.3}$$

i.e. it is close to a stochastic advection equation. This equation has explicit characteristics given by (3.5) below, so that we expect

$$f_\tau(z)\approx f_0(z_\tau)\,.$$

This approximation was rigorously justified, relying on a cancellation of all singularities emerging from the calculation of $\partial_t f_t^{(\nu)}$ with Itô's formula. This is the content of the following Lemma 3.1, from [10, Proposition 2.10]. This is of special interest for the study of determinants because

$$\frac{\mathrm{d}}{\mathrm{d}\nu}\sum_k\log\left|\lambda_k^{(\nu)}(t)-z\right|=e^{\frac{t}{2}}\Re\left(f_t^{(\nu)}(z)\right). \tag{3.4}$$

Indeed, (3.3) and (3.4) will give, by integration over $0\leqslant\nu\leqslant 1$, the fact that (1.18) and (1.19) are very close, as mentioned in the outline of the proof.

Lemma 3.1. *There exists $C_0>0$ such that with $\varphi=e^{C_0(\log\log N)^2}$, for any $\nu\in[0,1]$, $\kappa>0$ (small) and $D>0$ (large), there exists $N_0(\kappa,D)$ so that for any $N\geqslant N_0$ we have*

$$\mathbb{P}\left(\left|f_t^{(\nu)}(z)-f_0^{(\nu)}(z_t)\right|<\frac{\varphi}{N\eta}\text{ for all }0<t<1\text{ and }z=E+\mathrm{i}\eta,\frac{\varphi}{N}<\eta<1,|E|<2-\kappa\right)\geqslant 1-N^{-D}.$$

In the above, z_t is given by

$$z_t=\frac{1}{2}\left(e^{\frac{t}{2}}\left(z+\sqrt{z^2-4}\right)+e^{-\frac{t}{2}}\left(z-\sqrt{z^2-4}\right)\right). \tag{3.5}$$

For $z = \mathrm{i}\eta_0$ (remember (1.14)), we have

$$z_t = \mathrm{i}\left(\eta_0 + \frac{t\sqrt{\eta_0^2+4}}{2}\right) + \mathrm{O}\left(t^2\right), \tag{3.6}$$

For N large enough we have $\varphi/N < \eta_0 < 1$, so that we can apply Lemma 3.1. Therefore, integrating both sides of (3.4), we have by Lemma 3.1 that with overwhelming probability,

$$\begin{aligned}\sum_k \left(\log|x_k(\tau)+\mathrm{i}\eta_0| - \log|y_k(\tau)+\mathrm{i}\eta_0|\right) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\nu}\sum_k \log\left|\lambda_k^{(\nu)}(\tau)-z\right|\mathrm{d}\nu \\ = e^{\frac{t}{2}}\Re\int_0^1 f_\tau^{(\nu)}(z)\mathrm{d}\nu &= e^{\frac{t}{2}}\Re\int_0^1\left(f_0^{(\nu)}(z_\tau)+\mathrm{O}\left(\frac{\varphi}{N\eta_0}\right)\right)\mathrm{d}\nu = e^{\frac{t}{2}}\Re\int_0^1 f_0^{(\nu)}(z_\tau)\,\mathrm{d}\nu + \mathrm{o}(1).\end{aligned}$$

More precisely, the above estimates hold with probability $1-N^{-D}$ for large enough N, with rigorous justification by Markov's inequality based on the large moments $\mathbb{E}((\int_0^1 \left(f_\tau^{(\nu)}(z_0) - f_0^{(\nu)}(z_\tau)\right)\mathrm{d}\nu)^{2p})$, which are bounded by Lemma 3.1. As a consequence, we have proved the following proposition.

Proposition 3.2. *Let $\varepsilon > 0$, $\tau = N^{-\varepsilon}$ and let z_τ be as in (3.5) with $z = \mathrm{i}\eta_0$. Then for any $\delta > 0$,*

$$\lim_{N\to\infty}\mathbb{P}\left(\left|\sum_k\left(\log|x_k(\tau)+\mathrm{i}\eta_0| - \log|y_k(\tau)+\mathrm{i}\eta_0|\right) - \sum_k\left(\log|x_k(0)+z_\tau| - \log|y_k(0)+z_\tau|\right)\right| > \delta\right) = 0.$$

4 Conclusion of the Proof

We will conclude the proof of Theorem 1.2 in the real symmetric case in two steps. The first step is to prove a Green's function comparison theorem, and the second is to establish Theorem 1.2 assuming Lemma A.1, proved in the Appendix.

4.1 Green's Function Comparison Theorem. In this section, we first use Lemma 4.1 to choose a $\tilde{W}_0$ so that $\tilde{W}_\tau$ given by (1.15) and initial condition $\tilde{W}_0$, matches W closely up to fourth moment. We will then prove Theorem 4.4, which by the result of Section 2, says that $\log|\det\tilde{W}_\tau|$ and $\log|\det W|$ have the same law as $N\to\infty$.

Lemma 4.1 (Lemma 6.5 in [21])**.** *Let m_3 and m_4 be two real numbers such that*

$$m_4 - m_3^2 - 1 \geqslant 0, \quad m_4 \leqslant C_2 \tag{4.1}$$

for some positive constant C_2. Let ξ^G be a Gaussian random variable with mean 0 and variance 1. Then for any sufficiently small $\gamma > 0$ (depending on C_2), there exists a real random variable ξ_γ, with subgaussian decay and independent of ξ^G such that the first four moments of

$$\xi' = (1-\gamma)^{\frac{1}{2}}\xi_\gamma + \gamma^{\frac{1}{2}}\xi^G$$

are $m_1(\xi') = 0$, $m_2(\xi') = 1$, $m_3(\xi') = m_3$, and

$$|m_4(\xi') - m_4| \leqslant C\gamma$$

for some C depending on C_2.

Now since $\tilde{W}_\tau$ is defined by independent Ornstein-Uhlenbeck processes in each entry, it has the same distribution as

$$e^{-\tau/2}\tilde{W}_0 + \sqrt{1-e^{-\tau}}W$$

where W is a GOE matrix independent of $\tilde{W}_0$. So choosing $\gamma = 1-e^{-\tau}$, Lemma 4.1 says we can find $\tilde{W}_0$ so that the first three moments of the entries of $\tilde{W}_\tau$ match the first three moments of the entries of W, and the fourth moments of the entries of each differ by $\mathrm{O}(\tau)$. Our next goal is to prove Theorem 4.4 which says that with $\tilde{W}_\tau$ constructed this way, if Theorem 1.2 holds for $\tilde{W}_\tau$, then it holds for W. We first introduce stochastic domination and state Theorem 4.3 which we will use in the proof.

Definition 4.2. *Let* $X = \left(X^N(u) : N \in \mathbb{N}, u \in U^N\right), Y = \left(Y^N(u) : N \in \mathbb{N}, u \in U^N\right)$ *be two families of non-negative random variables, where* U^N *is a possibly* N*-dependent parameter set. We say that* X *is stochastically dominated by* Y*, uniformly in* u*, if for every* $\varepsilon > 0$ *and* $D > 0$*, there exists* $N_0(\varepsilon, D)$ *such that*

$$\sup_{u \in U^N} \mathbb{P}\left[X^N(u) > N^{\varepsilon} Y^N(u)\right] \leqslant N^{-D}$$

for $N \geqslant N_0$*. Stochastic domination is always uniform in all parameters, such as matrix indices and spectral parameters, that are not explicitly fixed. We will use the notation* $X = O_{\prec}(Y)$ *or* $X \prec Y$ *for the above property.*

Theorem 4.3 (Theorem 2.1 in [22])**.** *Let* W *be a Wigner matrix satisfying (1.5). Fix* $\zeta > 0$ *and define the domain*

$$S = S_N(\zeta) := \left\{E + \mathrm{i}\eta \,:\, |E| \leqslant \zeta^{-1},\, N^{-1+\zeta} \leqslant \eta \leqslant \zeta^{-1}\right\}.$$

Then uniformly for $i, j = 1, \ldots, N$ *and* $z \in S$*, we have*

$$s(z) = m(z) + \mathrm{O}_{\prec}\left(\frac{1}{N\eta}\right),$$

$$G_{ij}(z) = (W - z)^{-1}_{ij} = m(z)\delta_{ij} + \mathrm{O}_{\prec}\left(\sqrt{\frac{\Im\left(m(z)\right)}{N\eta}} + \frac{1}{N\eta}\right).$$

Theorem 4.4. *Let* $F : \mathbb{R} \to \mathbb{R}$ *be smooth with compact support, and let* W *and* V *be two Wigner matrices satisfying (1.5) such that for* $1 \leqslant i, j \leqslant N$*,*

$$\mathbb{E}\left(w_{ij}^a\right) = \begin{cases} \mathbb{E}\left(v_{ij}^a\right) & a \leqslant 3 \qquad (4.2) \\ \mathbb{E}\left(v_{ij}^a\right) + \mathrm{O}(\tau) & a = 4, \qquad (4.3) \end{cases}$$

where τ *is as in (1.17). Further, let* c_N *be any deterministic sequence and define*

$$u_N(W) = \frac{\log|\det\left(W + \mathrm{i}\eta_0\right)| + c_N}{\sqrt{\log N}}.$$

where η_0 *is as in (1.14). Then*

$$\lim_{N \to \infty} \mathbb{E}\left(F\left(u_N(W)\right) - F\left(u_N(V)\right)\right) = 0. \tag{4.4}$$

Proof. As in [54], where the authors also used the following technique to analyze fluctuations of determinants, we show that the effect of substituting W_{ij} in place of V_{ij} in V is negligible enough that making N^2 replacements, we conclude the theorem.

Fix (i, j) and let $E^{(ij)}$ be the matrix whose elements are $E^{(ij)}_{kl} = \delta_{ik}\delta_{jl}$. Let W_1 and W_2 be two adjacent matrices in the swapping process described above. Since W_1, W_2 differ in just the (i, j) and (j, i) coordinates, we may write

$$W_1 = Q + \frac{1}{\sqrt{N}} U, \qquad W_2 = Q + \frac{1}{\sqrt{N}} \tilde{U}$$

where Q is a matrix with $Q_{ij} = Q_{ji} = 0$, and

$$U = u_{ij} E^{(ij)} + u_{ji} E^{(ji)} \qquad \tilde{U} = \tilde{u}_{ij} E^{(ij)} + \tilde{u}_{ji} E^{(ji)}.$$

Importantly $U, \tilde{U}$ satisfy the same moment matching conditions we have imposed on $\tilde{W}_\tau$ and W. Now by the fundamental theorem of calculus, we have for any symmetric matrix W,

$$\log|\det(W + \mathrm{i}\eta_0)| = \sum_{k=1}^{N} \log|x_k + \mathrm{i}\eta_0| = \log|\det(W + \mathrm{i})| - N\,\Im \int_{\eta_0}^{1} s_W\left(\mathrm{i}\eta\right) \mathrm{d}\eta. \tag{4.5}$$

From the central limit theorems for linear statistics of Wigner matrices on macroscopic scales [40], $(\log|\det(W+\mathrm{i})| - \mathbb{E}(\log|\det(W+\mathrm{i})|))/\sqrt{\log N}$ converges to 0 in probability (the same result holds with W replaced with V), and from Lemma A.1 (which clearly holds with 1 in place of τ), $(\mathbb{E}(\log|\det(W+\mathrm{i})|) - \mathbb{E}(\log|\det(V+\mathrm{i})|))/\sqrt{\log N} \to 0$. Therefore (4.4) is equivalent to

$$\lim_{N\to\infty} \mathbb{E}\left(\widetilde{F}\left(N\Im\int_{\eta_0}^1 s_W(\mathrm{i}\eta)\,\mathrm{d}\eta\right) - \widetilde{F}\left(N\Im\int_{\eta_0}^1 s_V(\mathrm{i}\eta)\,\mathrm{d}\eta\right)\right) = 0, \tag{4.6}$$

where

$$\widetilde{F}(x) = F\left(\frac{\mathbb{E}(\log|\det(W+\mathrm{i})|) + c_N - x}{\sqrt{\log N}}\right).$$

We now expand s_{W_1} and s_{W_2} around s_Q, and then to Taylor expand $\widetilde{F}$. So let

$$R = R(z) = (Q-z)^{-1} \text{ and } S = S(z) = (W_1 - z)^{-1}.$$

By the resolvent expansion

$$S = R - N^{-1/2}RUR + \ldots + N^{-2}(RU)^4R - N^{-5/2}(RU)^5S,$$

we can write

$$N\int_{\eta_0}^1 s_{W_1}(\mathrm{i}\eta)\mathrm{d}\eta = \int_{\eta_0}^1 \mathrm{Tr}\,(S(\mathrm{i}\eta))\,\mathrm{d}\eta = \int_{\eta_0}^1 \mathrm{Tr}\,(R(\mathrm{i}\eta))\,\mathrm{d}\eta + \left(\sum_{m=1}^4 N^{-m/2}\hat{R}^{(m)}(\mathrm{i}\eta) - N^{-5/2}\Omega\right) := \hat{R} + \xi$$

where

$$\hat{R}^{(m)} = (-1)^m \int_{\eta_0}^1 \mathrm{Tr}\,((R(\mathrm{i}\eta)U)^m R(\mathrm{i}\eta))\,\mathrm{d}\eta \quad \text{and} \quad \Omega = \int_{\eta_0}^1 \mathrm{Tr}\left((R(\mathrm{i}\eta)U)^5 S(\mathrm{i}\eta)\right)\mathrm{d}\eta.$$

This gives us an expansion of s_{W_1} around s_Q. Now Taylor expand $\widetilde{F}(\hat{R}+\xi)$ as

$$\widetilde{F}\left(\hat{R}+\xi\right) = \widetilde{F}\left(\hat{R}\right) + \widetilde{F}'\left(\hat{R}\right)\xi + \ldots + \widetilde{F}^{(5)}\left(\hat{R}+\xi'\right)\xi^5 = \sum_{m=0}^5 N^{-m/2}A^{(m)} \tag{4.7}$$

where $0 < \xi' < \xi$, and we have introduced the notation $A^{(m)}$ in order to arrange terms according to powers of N. For example

$$A^{(0)} = \widetilde{F}\left(\hat{R}\right), \quad A^{(1)} = \widetilde{F}'\left(\hat{R}\right)\hat{R}^{(1)}, \quad A^{(2)} = \widetilde{F}'\left(\hat{R}\right)\hat{R}^{(2)} + \widetilde{F}''\left(\hat{R}\right)\left(\hat{R}^{(1)}\right)^2.$$

Making the same expansion for W_2, we record our two expansions as

$$\widetilde{F}\left(\hat{R}+\xi_i\right) = \sum_{m=0}^5 N^{-m/2}A_i^{(m)}, \quad i = 1,2,$$

with ξ_i corresponding to W_i. With this notation, we have

$$\mathbb{E}\left(\widetilde{F}\left(\hat{R}+\xi_1\right)\right) - \mathbb{E}\left(\widetilde{F}\left(\hat{R}+\xi_2\right)\right) = \mathbb{E}\left(\sum_{m=0}^5 N^{-m/2}\left(A_1^{(m)} - A_2^{(m)}\right)\right).$$

Now only the first three moments of $U, \tilde{U}$ appear in the terms corresponding to $m = 1,2,3$, so by the moment matching assumption (4.2), all of these terms are all identically zero. Next, consider $m = 4$. Every term with first, second, and third moments of U and $\tilde{U}$ is again zero, and what remains is

$$\mathbb{E}\left(\widetilde{F}'(\hat{R})\left(\hat{R}_1^{(4)} - \hat{R}_2^{(4)}\right)\right).$$

So we can discard $A^{(4)}$ if

$$\int_{\eta_0}^{1}\left|\mathbb{E}\left(\mathrm{Tr}\left((RU)^4R\right)-\mathrm{Tr}\left((R\tilde{U})^4R\right)\right)\right|\mathrm{d}\eta \tag{4.8}$$

is small. To see that this is in fact the case, we expand the traces, and apply Theorem 4.3 along with our fourth moment matching assumption (4.3). Specifically,

$$\mathrm{Tr}\left((RU)^4R\right)=\sum_j\left(\sum_{i_1,\dots,i_8}R_{ji_1}U_{i_1i_2}R_{i_2i_3}\dots U_{i_7i_8}R_{i_8j}\right).$$

Writing the corresponding Tr for W_2 and applying the moment matching assumption, we see that we can bound (4.8) by

$$\mathrm{O}(\tau)\int_{\eta_0}^{1}\sum_j\sum_{i_1,\dots,i_8}\mathbb{E}\left(|R_{ji_1}R_{i_2i_3}R_{i_4i_5}R_{i_6i_7}R_{i_8j}|\right)\mathrm{d}\eta,$$

where the sums over $i_1,\dots,i_8$ (above and below) are just sums over p,q, with p,q are the indices such that $U_{pq},\tilde{U}_{pq}$ and $U_{qp},\tilde{U}_{qp}$ are non zero. To bound the terms in the sum, we need to count the number of diagonal and off-diagonal terms in each product. When $j\notin\{p,q\}$, R_{ji_1} and R_{i_8j} are certainly off-diagonal entries of R. Applying Cauchy-Schwartz, we obtain that for any $\gamma>0$,

$$\mathrm{O}(\tau)\int_{\eta_0}^{1}\sum_{j\notin\{p,q\}}\sum_{i_1,\dots,i_8}\mathbb{E}\left(|R_{ji_1}R_{i_2i_3}R_{i_4i_5}R_{i_6i_7}R_{i_8j}|\right)\mathrm{d}\eta=\mathrm{O}\left(\tau N^{1+2\gamma}\int_{\eta_0}^{1}\frac{1}{N\eta}\mathrm{d}\eta\right)=\mathrm{O}\left(N^{2\gamma-\varepsilon}\log(N)\right).$$

Similarly,

$$\mathrm{O}(\tau)\int_{\eta_0}^{1}\sum_{j\in\{p,q\}}\sum_{i_1,\dots,i_8}\mathbb{E}\left(|R_{ji_1}R_{i_2i_3}R_{i_4i_5}R_{i_6i_7}R_{i_8j}|\right)\mathrm{d}\eta=\mathrm{O}\left(\tau N^{\varepsilon/2}\right)=\mathrm{O}\left(N^{-\varepsilon/2}\right).$$

Since $A^{(4)}$ has a pre-factor of N^{-2} in (4.7), and the above holds for every choice of $\gamma>0$, in our entire entry swapping scheme starting from V and ending with W, the corresponding error is $\mathrm{o}(1)$.

Lastly we comment on the error term $A^{(5)}$. All terms in $A^{(5)}$ not involving Ω can be dealt with as above. The only term involving Ω is $\widetilde{F}'(\hat{R})\Omega$, and to deal with this, we can expand the expression for Ω as above. We do not have any moment matching condition for the fifth moments of $U,\tilde{U}$, but (1.5) means that their fifth moments are bounded which is enough for our purpose since $A^{(5)}$ has a pre-factor of $N^{-5/2}$ above. □

4.2 Proof of Theorem 1.2. In this section we first prove Proposition 4.5 and, using Lemma A.1, we conclude the proof of Theorem 1.2.

Proposition 4.5. *Recall $\tau=N^{-\varepsilon}$. There exist ε_0,C such that for any fixed $0<\varepsilon<\varepsilon_0$, for large enough N, we have*

$$\mathrm{Var}\left(\sum_k\log|x_k(0)+\mathrm{i}\tau|\right)\leqslant C(1+\varepsilon\log N).$$

Proof. We outline two proofs, which are trivial extensions of existing linear statistics asymptotics on global scales, to the case of almost macroscopic scales. The tool for this extension is the rigidity estimate from [22]: for any $c,D>0$, there exists N_0 such that for any $N\geqslant N_0$ and $k\in[\![1,N]\!]$ we have

$$\mathbb{P}\left(|x_k-\gamma_k|>N^{-\frac{2}{3}+c}\min(k,N+1-k)^{-\frac{1}{3}}\right)\leqslant N^{-D}. \tag{4.9}$$

For the first proof, we use (4.9) to bound all the error terms in the proof of [40, Theorem 3.6] (these error terms all depend on [40, Theorem 3.5], which can be improved via (4.9) to $\mathrm{Var}(u_N(t))\leqslant N^c(1+|t|)$ and

$\mathrm{Var}\,(\mathcal{N}_N(\varphi)) \leqslant N^c \|\varphi\|_{\mathrm{Lip}}^2$). What we obtain is that if φ (possibly depending on N) satisfies $\int |t|^{100}\hat\varphi(t) < N^{1/100}$, then $\sum \varphi(x_k) - \mathbb{E}(\sum \varphi(x_k))$ has limiting variance asymptotically equivalent to

$$V_{\mathrm{Wig}}[\varphi] = \frac{1}{2\pi^2}\int_{(-2,2)^2}\left(\frac{\Delta\varphi}{\Delta\lambda}\right)^2 \frac{4-\lambda_1\lambda_2}{\sqrt{4-\lambda_1^2}\sqrt{1-\lambda_2^2}}\,\mathrm{d}\lambda_1\mathrm{d}\lambda_2 + \frac{\kappa_4}{2\pi^2}\left(\int_{-2}^{2}\varphi(\mu)\frac{2-\mu^2}{\sqrt{4-\mu^2}}\,\mathrm{d}\mu\right)^2, \tag{4.10}$$

where $\Delta\varphi = \varphi(\lambda_1) - \varphi(\lambda_2)$, $\Delta\lambda = \lambda_1 - \lambda_2$, $\mu_4 = \mathbb{E}\left(W_{jk}^4\right)$, $\kappa_4 = \mu_4 - 3$ is the fourth cumulant of the off-diagonal entries of W. We choose $\varphi(x) = \varphi_N(x) = \frac{1}{2}\log(x^2+\tau^2)\chi(x)$ with χ fixed, smooth, compactly supported, equal to 1 on $(-3,3)$. Note that for ε_0 small enough, we have $\int |t|^{100}\hat\varphi(t) < N^{1/100}$. Then by (4.9) and (4.10),

$$V_{\mathrm{Wig}}[\log|\cdot - \mathrm{i}\tau|] \sim V_{\mathrm{Wig}}[\varphi] \leqslant C\iint\left(\frac{\Delta\varphi}{\Delta\lambda}\right)^2 \mathrm{d}\lambda_1\mathrm{d}\lambda_2 = C\int |\xi|\,|\hat\varphi(\xi)|^2\,\mathrm{d}\xi,$$

and the above right hand side can be bounded as follows. We have

$$|\hat\varphi_N(\xi)| = \left|\frac{1}{2\pi}\int_{\mathbb{R}}\varphi_N(x)e^{-i\xi x}\,\mathrm{d}x\right| \leqslant C\left|\int_{-5}^{5}\frac{x}{x^2+\tau^2}\frac{e^{-i\xi x}}{i\xi}\,\mathrm{d}x\right| = C\left|\frac{1}{\xi}\int_0^{5/\tau}\frac{x}{x^2+1}\sin(x\xi\tau)\,\mathrm{d}x\right|.$$

For $0<\xi<5$, the inequality $|\sin x| < x$ shows $|\hat\varphi_N(\xi)| = \mathrm{O}(1)$, and when $\xi > 5/\tau$, integration by parts shows $|\hat\varphi_N(\xi)| = \mathrm{O}\left(\frac{1}{\xi^2\tau}\right)$. When $5 < \xi < 5/\tau$, first note

$$\int_0^{\frac{5}{\tau}}\sin(\xi\tau x)\,\frac{x}{x^2+1}\,\mathrm{d}x = C + \int_1^{\frac{5}{\tau}}\frac{\sin(\xi\tau x)}{x}\,\mathrm{d}x = C + \int_{\xi\tau}^{1}\frac{\sin y}{y}\,\mathrm{d}y + \int_1^{5\xi}\frac{\sin y}{y}\,\mathrm{d}y.$$

Using $|\sin y| < |y|$, we see that the first term is $\mathrm{O}(1)$, and integrating by parts, we see that the second term is $\mathrm{O}(1)$ as well. This means

$$\int |\xi|\,|\hat\varphi_N(\xi)|^2\,\mathrm{d}\xi \leqslant C + C\int_5^{\frac{5}{\tau}}\frac{1}{\xi}\,\mathrm{d}\xi = \mathrm{O}\left(1+|\log\tau|\right),$$

which concludes the proof.

The second proof is similar but more direct. Theorem 3 in [34] implies that for $z_1 = \mathrm{i}\eta_1, z_2 = \mathrm{i}\eta_2$ at macroscopic distance from the real axis, and $\eta_1 = \mathrm{Im}\,z_1 > 0, \eta_2 = \mathrm{Im}\,z_2 < 0$, we have

$$\left|\mathrm{Cov}\left(\sum_k\frac{1}{z_1-x_k},\sum_k\frac{1}{z_2-x_k}\right)\right| \leqslant \frac{C}{(\eta_1-\eta_2)^2} + f(z_1,z_2) + \mathrm{O}(N^{-1/2}),$$

where f is a function uniformly bounded on any compact subset of $\mathbb{C}^2$. Using (4.9), one easily obtains that the formula above holds uniformly with $|\mathrm{Im}\,z_1|, |\mathrm{Im}\,z_2| > N^{-1/10}$, and the deteriorated error term $\mathrm{O}(N^{-1/10})$, for example. Note that

$$\log|\det(W+\mathrm{i}\eta)| = \log|\det(W+\mathrm{i})| - N\,\Im\int_\eta^1 s_W(\mathrm{i}x)\,\mathrm{d}x.$$

and $\log|\det(W+\mathrm{i})|$ has fluctuations of order 1 due to the above macroscopic central limit theorems. For for $\eta > N^{-1/10}$, the variance of the above integral can be bounded by $\iint_{[\eta,1]^2}\frac{1}{|\eta_1+\eta_2|^2}\,\mathrm{d}\eta_1\mathrm{d}\eta_2 \leqslant C|\log\eta|$, which concludes the proof. □

From (1.4) and Proposition 2.1, for some explicit deterministic c_N we have

$$\frac{\sum_{k=1}^N \log|x_k(\tau)+\mathrm{i}\eta_0| + c_N}{\sqrt{\log N}} \to \mathscr{N}(0,1), \tag{4.11}$$

and Proposition 3.2 implies that

$$\frac{\sum_{k=1}^N \log|y_k(\tau)+\mathrm{i}\eta_0|+c_N}{\sqrt{\log N}}+\frac{\sum_{k=1}^N \log|x_k(0)+z_\tau|-\sum_{k=1}^N \log|y_k(0)+z_\tau|}{\sqrt{\log N}}\to \mathscr{N}(0,1).$$

Lemma A.1 and Proposition 4.5 show that the second term above, call it X, satisfies $\mathbb{E}(X^2)<C\varepsilon$, for some universal C. Thus for any fixed smooth and compactly supported function F,

$$\begin{aligned}\mathbb{E}\left(F\left(\frac{\sum_{k=1}^N \log|y_k(\tau)+\mathrm{i}\eta_0|+c_N}{\sqrt{\log N}}\right)\right)&=\mathbb{E}\left(F\left(\frac{\sum_{k=1}^N \log|x_k(\tau)+\mathrm{i}\eta_0|+c_N}{\sqrt{\log N}}+X\right)\right)+\mathrm{O}\left(\|F\|_{\mathrm{Lip}}(\mathbb{E}\left(X^2\right))^{1/2}\right)\\&=\mathbb{E}\left(F(\mathscr{N}(0,1))\right)+\mathrm{o}(1)+\mathrm{O}\left(\varepsilon^{1/2}\right).\end{aligned}$$

With Theorem 4.4, the above equation implies

$$\mathbb{E}\left(F\left(\frac{\log|\det(W+\mathrm{i}\eta_0)|+c_N}{\sqrt{\log N}}\right)\right)=\mathbb{E}\left(F(\mathscr{N}(0,1))\right)+\mathrm{o}(1)+\mathrm{O}\left(\varepsilon^{1/2}\right),$$

and by Proposition 2.1, we obtain

$$\mathbb{E}\left(F\left(\frac{\log|\det W|+\frac{N}{2}}{\sqrt{\log N}}\right)\right)=\mathbb{E}\left(F(\mathscr{N}(0,1))\right)+\mathrm{o}(1)+\mathrm{O}\left(\varepsilon^{1/2}\right). \tag{4.12}$$

Since ε is arbitrarily small, this concludes the proof.

Appendix A: Expectation of Regularized Determinants

We prove the following result, which we use both in the proof of Theorem 4.4, and to conclude the proof of Theorem 1.2.

Lemma A.1. *Recall the notation $\tau=N^{-\varepsilon}$, and let $\{x_k\}_{k=1}^N$, $\{y_k\}_{k=1}^N$ denote the eigenvalues of two Wigner matrices, W_1 and W_2. Then*

$$\mathbb{E}\left(\sum_k \log|x_k+\mathrm{i}\tau|-\sum_k \log|y_k+\mathrm{i}\tau|\right)=\mathrm{O}(1).$$

Proof. By the fundamental theorem of calculus, we can write

$$\sum_k \log|x_k+\mathrm{i}\tau|=\sum_{k=1}^N \log\left|x_k+\mathrm{i}N^\delta\right|+N\int_\tau^{N^\delta}\Im\left(s_{W_1}(\mathrm{i}\eta)\right)\mathrm{d}\eta \tag{A.1}$$

with s_W as in (1.20), and $\delta>0$. Writing the same expression for W_2 and taking the difference, we first note that by (4.9), we have that for any $\gamma>0$,

$$\mathbb{E}\left(\left|\sum_{k=1}^N\left(\log\left|x_k+\mathrm{i}N^\delta\right|-\log\left|y_k+\mathrm{i}N^\delta\right|\right)\right|\right)\leqslant \mathbb{E}\left(N^{-\delta}\sum_{i=1}^N|x_k-y_k|\right)=\mathrm{O}\left(N^{\gamma-\delta}\right). \tag{A.2}$$

Therefore, we only need to bound

$$\Im\left(N\int_\tau^{N^\delta}\mathbb{E}\left(s_{W_1}(\mathrm{i}\eta)-s_{W_2}(\mathrm{i}\eta)\right)\mathrm{d}\eta\right). \tag{A.3}$$

Let $z=E+\mathrm{i}\eta$ be in $S\left(\frac{1}{100}\right)$ (as defined in Theorem 4.3), and define

$$f(z)=N\left(s_{W_1}(z)-s_{W_2}(z)\right).$$

We will first estimate $\mathbb{E}(f(z))$ for $\tau < \eta < 5$, where we can use Theorem 4.3. Then we will use complex analysis to extend this estimate to $5 < \eta < N^{\delta}$.

Let $\tau < \eta < 5$. Following the notation of [22], let W be a Wigner matrix and let

$$v_i = G_{ii} - m_{sc}, \quad [v] = \frac{1}{N}\sum_{i=1}^{N} v_i, \quad G(z) = (W-z)^{-1},$$

We will use the notation $W^{(i)}$ to denote the $(N-1)\times(N-1)$ matrix obtained by removing the i^{th} row and column from W, and w_i to denote the i^{th} column of $W^{(i)}$ without W_{ii}. We will also denote the eigenvalues of W by $\lambda_1 < \lambda_2 < \dots \lambda_N$. Let $G^{(i)} = \left(W^{(i)} - z\right)^{-1}$. Applying the Schur complement formula to W (see Lemma 4.1 in [21]), we have

$$v_i + m_{sc} = \left(-z - m_{sc} + W_{ii} - [v] + \frac{1}{N}\sum_{j\neq i}\frac{G_{ij}G_{ji}}{G_{ii}} - Z_i\right)^{-1} = \left(-z - m_{sc} - ([v] - \Gamma_i)\right)^{-1} \tag{A.4}$$

where

$$Z_i = (1-\mathbb{E}_i)(w_i, G^{(i)}w_i), \quad \mathbb{E}_i(X) = \mathbb{E}\left(X|W^{(i)}\right), \quad \Gamma_i = \frac{1}{N}\sum_{j\neq i}\frac{G_{ij}G_{ji}}{G_{ii}} - Z_i + W_{ii}.$$

By Theorem 4.3,

$$|\Gamma_i - [v]| = \mathrm{O}_{\prec}\left(\frac{1}{N^{\frac{1}{2}}\eta^{\frac{1}{2}}}\right), \tag{A.5}$$

so we can expand (A.4) around $-z - m_{sc}$. Using (1.22), we find

$$\begin{aligned} v_i &= m_{sc}^2\left([v]-\Gamma_i\right) + m_{sc}^3\left([v]-\Gamma_i\right)^2 + \mathrm{O}\left(\left([v]-\Gamma_i\right)^3\right) \\ &= m_{sc}^2\left([v] - W_{ii} - \frac{1}{N}\sum_{j\neq i}\frac{G_{ij}G_{ji}}{G_{ii}} + Z_i\right) + m_{sc}^3\left([v]-\Gamma_i\right)^2 + \mathrm{O}\left(\left([v]-\Gamma_i\right)^3\right), \end{aligned}$$

and summing over i and taking expectation, we have

$$\mathbb{E}\left((1-m_{sc}^2)\sum_i v_i\right) = \mathbb{E}\left(-\frac{m_{sc}^2}{N}\sum_{i=1}^{N}\sum_{j\neq i}^{N}\frac{G_{ij}G_{ji}}{G_{ii}} + m_{sc}^3\sum_i\left([v]-\Gamma_i\right)^2 + \sum_i \mathrm{O}\left(\left([v]-\Gamma_i\right)^3\right)\right), \tag{A.6}$$

since the expectations of W_{ii} and Z_i are both zero. We now use this expansion to estimate $\mathbb{E}(f(z))$. Since we $\tau < \eta < 5$, we have by Theorem 4.3 that

$$\frac{m_{sc}^2}{N}\sum_i\sum_{j\neq i}\frac{G_{ij}G_{ji}}{G_{ii}} = \frac{m_{sc}}{N}\left(\sum_{i,j=1}^{N}G_{ij}G_{ji} - \sum_{i=1}^{N}(G_{ii})^2\right) + \mathrm{O}_{\prec}\left(\frac{1}{N^{\frac{1}{2}}\eta^{\frac{1}{2}}}\right)\frac{m_{sc}}{N}\sum_i\sum_{j\neq i}\left|G_{ij}G_{ji}\right|. \tag{A.7}$$

Now observe that

$$\frac{m_{sc}}{N}\sum_{i,j}G_{ij}G_{ji} = \frac{m_{sc}}{N}\mathrm{Tr}\left(G^2\right) = \frac{m_{sc}}{N}\sum_{k=1}^{N}\frac{1}{(\lambda_k - z)^2},$$

and

$$\frac{1}{N}\sum_{k=1}^{N}\frac{1}{(x_k-z)^2} - \frac{1}{N}\sum_{k=1}^{N}\frac{1}{(y_k-z)^2} = s'_{W_1}(z) - s'_{W_2}(z).$$

Choosing $\mathcal{C}(z) = \left\{w : |w-z| = \frac{\eta}{2}\right\}$, we have

$$\left|s'_{W_1}(z) - s'_{W_2}(z)\right| \leqslant \frac{1}{2\pi}\int_{\mathcal{C}(z)}\frac{|s_{W_1}(z) - s_{W_2}(z)|}{(\zeta - z)^2}\,\mathrm{d}\zeta = \mathrm{O}_{\prec}\left(\frac{1}{N\eta^2}\right) \tag{A.8}$$

by Theorem 4.3. Again applying Theorem 4.3, we have

$$\frac{m_{sc}}{N}\sum_{i=1}^{N}(G_{ii})^2=\frac{m_{sc}}{N}\sum_{i=1}^{N}(v_i+m_{sc})^2=m_{sc}^3+\mathrm{O}_\prec\left(\frac{1}{N\eta}\right),\text{ and }\sum_{i\neq j}|G_{ij}G_{ji}|=\mathrm{O}_\prec\left(\frac{1}{\eta}\right).$$

Putting together these estimates we have

$$\mathbb{E}\left(\int_\tau^5\sum_{i=1}^{N}\sum_{j\neq i}^{N}\frac{m_{sc}^2}{N\left(1-m_{sc}^2\right)}\left(\frac{(G_1)_{ij}(G_1)_{ji}}{(G_1)_{ii}}-\frac{(G_2)_{ij}(G_2)_{ji}}{(G_2)_{ii}}\right)\mathrm{d}\eta\right)=\mathbb{E}\left(\int_\tau^5\mathrm{O}_\prec\left(\frac{1}{N^{\frac{1}{2}}\eta}\right)\mathrm{d}\eta\right)=\mathrm{o}(1).$$

Next, consider

$$m_{sc}^3\sum_{i=1}^{N}\left([v]-\Gamma_i\right)^2=m_{sc}^3\sum_{i=1}^{N}\left([v]^2-2[v]\Gamma_i+\Gamma_i^2\right). \tag{A.9}$$

By Theorem 4.3, $[v]=\mathrm{O}_\prec\left(\frac{1}{N\eta}\right)$, so summing over i and integrating with respect to η, we find

$$\mathbb{E}\left(\int_\tau^5\sum_i\frac{m_{sc}^3}{1-m_{sc}^2}[v]^2\mathrm{d}\eta\right)=\mathbb{E}\left(\int_\tau^5\mathrm{O}_\prec\left(\frac{1}{N\eta^{\frac{5}{2}}}\right)\right)=\mathrm{O}\left(\frac{N^{\frac{3\varepsilon}{2}+\gamma}}{N}\right)$$

for any $\gamma>0$. Next, we estimate $\mathbb{E}\left(m_{sc}^3\sum_i\Gamma_i^2\right)$. Expanding Γ_i^2, we have

$$\Gamma_i^2=W_{ii}^2+\left(\frac{1}{N}\sum_{j\neq i}\frac{G_{ij}G_{ji}}{G_{ii}}\right)^2+Z_i^2+2\left(\frac{W_{ii}}{N}\sum_{j\neq i}\frac{G_{ij}G_{ji}}{G_{ii}}-W_{ii}Z_i-\frac{Z_i}{N}\sum_{j\neq i}\frac{G_{ij}G_{ji}}{G_{ii}}\right). \tag{A.10}$$

By definition, we have $\mathbb{E}\left(W_{ii}^2\right)=\frac{1}{N}$. Therefore $\mathbb{E}\left((W_1)_{ii}^2-(W_2)_{ii}^2\right)=0$, and by Theorem 4.3, we have

$$\sum_{i=1}^{N}m_{sc}^3\left(\frac{1}{N}\sum_{j\neq i}\frac{G_{ij}G_{ji}}{G_{ii}}\right)^2=\mathrm{O}_\prec\left(\frac{1}{N\eta^2}\right).$$

Next, we examine $\mathbb{E}\left(\sum_{i=1}^{N}Z_i^2\right)$. Note that by the independence of $w_i(l)$ and $w_i(k)$ and the independence of w_i and $G^{(i)}$, we have

$$\mathbb{E}_i\left(\left\langle w_i,G^{(i)}w_i\right\rangle\right)=\mathbb{E}_i\left(\sum_{k,l}G_{kl}^{(i)}w_i(l)\overline{w_i(k)}\right)=\mathbb{E}_i\left(\sum_{k=1}^{N}G_{kk}^{(i)}\overline{w_i^2(k)}\right)=\frac{1}{N}\mathrm{Tr}\left(G^{(i)}\right).$$

Therefore,

$$\mathbb{E}\left(\sum_{i=1}^{N}Z_i^2\right)=\sum_{i=1}^{N}\mathbb{E}_{W^{(i)}}\left(\mathbb{E}_i\left(\left(\left\langle w_i,G^{(i)}w_i\right\rangle^2\right)-\left(\frac{1}{N}\mathrm{Tr}\left(G^{(i)}\right)\right)^2\right)\right). \tag{A.11}$$

Expanding the first term on the left hand side above, we have

$$\mathbb{E}_i\left(\left\langle w_i,G^{(i)}w_i\right\rangle^2\right)=\mathbb{E}_i\left(\sum_{k,l,k',l'}G_{kl}^{(i)}w_i(l)\overline{w_i(k)}G_{k'l'}^{(i)}w_i(l')\overline{w_i(k')}\right). \tag{A.12}$$

The only terms which contribute to this sum are those for which at least two pairs of the indices amongst k,k',l,l' coincide. Consider first the case $k=l$, $k'=l'$, $k\neq k'$. The contribution of these terms to the above sum is

$$\mathbb{E}_i\left(\sum_{k\neq l}G_{kk}^{(i)}G_{ll}^{(i)}\left|w_i(k)\right|^2\left|w_i(l)\right|^2\right)=\left(\frac{1}{N}\mathrm{Tr}\left(G^{(i)}\right)\right)^2-\frac{1}{N^2}\sum_{k=1}^{N}\left(G_{kk}^{(i)}\right)^2.$$

The first term on the right hand side here cancels the second term on the right hand side of (A.11). For the second term, by Theorem 4.3, we have

$$\frac{1}{N^2}\sum_{i=1}^{N}\sum_{k=1}^{N}\left(\left(G_1^{(i)}\right)^2_{kk}-\left(G_2^{(i)}\right)^2_{kk}\right)=\mathrm{O}_{\prec}\left(\frac{1}{N^{\frac{1}{2}}\eta^{\frac{1}{2}}}\right). \tag{A.13}$$

Next consider the case where $k=k'$, $l=l'$, $k\neq l$. We consider separately the case where W has real entries, and the case where W has complex entries. In the first case, we can assume that the eigenvectors of W have real entries. Therefore, by the spectral decomposition of G, we have

$$\frac{1}{N^2}\sum_{i=1}^{N}\sum_{k\neq l}\left(G_{kl}^{(i)}\right)^2=\frac{1}{N^2}\sum_{i=1}^{N}\left(\sum_{k,l}\left(G_{kl}^{(i)}\right)^2-\sum_{k\neq i}\left(G_{kk}^{(i)}\right)^2\right)=\frac{1}{N^2}\sum_{i=1}^{N}\sum_{k\neq i}\left(\frac{1}{\left(\lambda_k^{(i)}-z\right)^2}-\left(G_{kk}^{(i)}\right)^2\right).$$

Using (A.8) and (A.13), this gives us

$$\frac{1}{N^2}\sum_{i=1}^{N}\sum_{k\neq l}\left(\left(G_1^{(i)}\right)^2_{kl}-\left(G_2^{(i)}\right)^2_{kl}\right)=\mathrm{O}_{\prec}\left(\frac{1}{N^{\frac{1}{2}}\eta^{2}}\right).$$

If instead W has complex entries, this term is identically zero. Indeed the corresponding expression becomes

$$\sum_{i=1}^{N}\sum_{k\neq l}\left(G_{kl}^{(i)}\right)^2\mathbb{E}_i\left(\left(\overline{w_i(k)}\right)^2(w_i(l))^2\right),$$

and because we have assumed that that for $i\neq j$, W_{ij} is of the form $x+\mathrm{i}y$ where $\mathbb{E}(x)=\mathbb{E}(y)=0$ and $\mathbb{E}\left(x^2\right)=\mathbb{E}\left(y^2\right)$, we have $\mathbb{E}\left(W_{ij}\right)^2=0$. There remain two cases to consider. Suppose $k'=l$, $l'=k$, $k\neq l$. Then

$$\sum_{i=1}^{N}\mathbb{E}_i\left(\sum_{k\neq l}G_{kl}^{(i)}G_{lk}^{(i)}\left|w_i(k)\right|^2\left|w_i(l)\right|^2\right)=\sum_{i}\frac{1}{N^2}\left(\sum_{k,l}G_{kl}^{(i)}G_{lk}^{(i)}-\sum_{k=1}^{N}\left(G_{kk}^{(i)}\right)^2\right),$$

and we may estimate the difference of this expression at G_1 and G_2 as we did the first term on the right hand side of (A.7). Lastly, we consider the case $k=k'=l=l'$. By Definition 1.1 and Theorem 4.3, there exists a constant C such that

$$\sum_{i=1}^{N}\mathbb{E}_i\left(\sum_{k=1}^{N}\left(G_{kk}^{(i)}\right)^2\left|w_i(k)\right|^4\right)=Cm_{sc}^2(z)+\mathrm{O}_{\prec}\left(\frac{1}{N^{\frac{1}{2}}\eta^{\frac{1}{2}}}\right). \tag{A.14}$$

Therefore

$$\sum_{i=1}^{N}\mathbb{E}_i\left(\sum_{k=1}^{N}\left(G_1^{(i)}\right)^2_{kk}\left|w_i^{(1)}(k)\right|^4-\left(G_2^{(i)}\right)^2_{kk}\left|w_i^{(2)}(k)\right|^4\right)=Cm_{sc}^2(z)+\mathrm{O}_{\prec}\left(\frac{1}{N^{\frac{1}{2}}\eta^{\frac{1}{2}}}\right).$$

In summary,

$$\mathbb{E}\left(\sum_{i=1}^{N}\left[(Z_1)_i^2-(Z_2)_i^2\right]\right)=\mathrm{O}(1). \tag{A.15}$$

Returning to (A.10), by Theorem 4.3 we have

$$\mathbb{E}\left(\sum_{i=1}^{N}\frac{W_{ii}}{N}\sum_{j\neq i}\frac{G_{ij}G_{ji}}{G_{ii}}\right)\leqslant\sum_{i=1}^{N}\left(\left(\mathbb{E}\left(W_{ii}^2\right)\right)^{\frac{1}{2}}\left(\mathbb{E}\left(\frac{1}{N}\sum_{j\neq i}\frac{G_{ij}G_{ji}}{G_{ii}}\right)^2\right)^{\frac{1}{2}}\right)=\mathrm{O}\left(\frac{N^{\gamma}}{N^{\frac{1}{2}}\eta}\right)$$

for any $\gamma > 0$. We also have that $\mathbb{E}\left(W_{ii} Z_i\right) = 0$. To bound the remaining term in (A.10), we first note that using the same argument as we did to prove (A.15), we have

$$\mathbb{E}\left(|Z_i|^2\right) = \mathrm{O}\left(\frac{1}{N\eta}\right). \tag{A.16}$$

Applying Theorem 4.3, we therefore conclude that

$$\mathbb{E}\left(\left|\sum_{i=1}^{N} \frac{Z_i}{N} \sum_{j \neq i} \frac{G_{ij} G_{ji}}{G_{ii}}\right|\right) = \mathrm{O}\left(\frac{N^\gamma}{N\eta^2}\right),$$

for any $\gamma > 0$. Putting together all of our estimates concerning (A.10), we have

$$\mathbb{E}\left(\int_\tau^5 \sum_{k=1}^{N} \left(\frac{m_{sc}^3}{1 - m_{sc}^2} \Gamma_k^2\right) \mathrm{d}\eta\right) = \mathrm{O}(1), \tag{A.17}$$

where we used $\frac{m_{sc}^3}{1-m_{sc}^2} = \mathrm{O}(1)$. Returning to (A.9), by Cauchy-Schwarz and Theorem 4.3 we have that for any $\gamma > 0$

$$\mathbb{E}\left(\sum_{i=1}^{N} m_{sc}^3 [v] \Gamma_i\right) = \mathrm{O}\left(\frac{N^\gamma}{N^{\frac{1}{2}} \eta^{\frac{3}{2}}}\right).$$

In total, we have

$$\mathbb{E}\left(\int_\tau^5 \left(\frac{m_{sc}^3}{1 - m_{sc}^2}\right) \sum_{i=1}^{N} \left([v]^2 - 2[v]\Gamma_i + \Gamma_i^2\right) \mathrm{d}\eta\right) = \mathrm{O}\left(1\right). \tag{A.18}$$

Finally, we have

$$\int_\tau^5 \sum_i \left|[v] - \Gamma_i\right|^3 \mathrm{d}\eta = \mathrm{o}(1)$$

using (A.5).

In summary, we have proved that for $z = (E + \mathrm{i}\eta) \in S\left(\frac{1}{100}\right)$, and any $\gamma > 0$,

$$\mathbb{E}\left(f(z)\right) = \frac{C m_{sc}^5(z)}{1 - m_{sc}^2(z)} + \mathrm{O}\left(\frac{N^\gamma}{N^{\frac{1}{2}} \eta^{\frac{5}{2}}}\right). \tag{A.19}$$

In particular, this means that

$$\int_\tau^5 \mathbb{E}\left(f\left(\mathrm{i}\eta\right)\right) \mathrm{d}\eta = \mathrm{O}(1).$$

To complete the proof of this lemma, we need to estimate $\int_5^{N^\delta} \mathbb{E}\left(f\left(\mathrm{i}\eta\right)\right) \mathrm{d}\eta$. Let

$$q(z) = \mathbb{E}\left(f(z)\right), \quad \tilde{q}(z) = q\left(\frac{1}{z}\right).$$

The function q is clearly bounded as $|z| \to \infty$, so $\tilde{q}$ is bounded at 0, which by Riemann's theorem is therefore a removable singularity. By (4.9), this means

$$\mathbb{P}\left(\tilde{q}(z) \text{ is analytic in } \mathbb{C} \setminus \left\{\left(-\infty, -\frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)\right\}\right) \geqslant 1 - N^{-D},$$

and so with overwhelming probability, we can write

$$q(z) = \tilde{q}(w) = \frac{1}{2\pi\mathrm{i}} \int_{\mathcal{C}_\Gamma} \frac{\tilde{q}(\xi)}{\xi - w} \mathrm{d}\xi = -\frac{1}{2\pi\mathrm{i}} \int_{C_\gamma} \frac{q(\xi)}{\xi - w\xi} \mathrm{d}\xi \tag{A.20}$$

where $w = \frac{1}{z}$ and we choose $\mathcal{C}_\gamma = \{x + \mathrm{i}y : |x| = 4, |y| = 4\}$ so that w is inside C_Γ, and $\tilde{q}$ is analytic there. Now we can estimate the right hand side using (A.19) and (4.9). Since $\Im(z) > 5$, we have $\sup_{\xi\in C_\gamma} \frac{1}{|\xi - w\xi|} = \mathrm{O}(1)$. Furthermore, for $z \in [4 - \mathrm{i}\tau, 4 + \mathrm{i}\tau]$, by (4.9) we have

$$|f(z)| = \left|\sum_{k=1}^{N}\left(\frac{1}{x_k - z} - \frac{1}{y_k - z}\right)\right| = \mathrm{O}_\prec(1).$$

Therefore, using (A.19), when $|\Im(z)| > 5$, for any $\gamma > 0$, we have,

$$|q(z)| \leqslant \sup_{\xi\in C_\gamma} \frac{1}{|\xi - w\xi|}\mathrm{O}\left(\int_{-4}^{4} \frac{N^\gamma}{N^{\frac{1}{2}}}\mathrm{d}x + \int_{\tau}^{4} \frac{N^\gamma}{N^{\frac{1}{2}}y^{\frac{5}{2}}}\mathrm{d}y + \int_0^\tau N^\gamma\,\mathrm{d}y\right) = \mathrm{O}\left(N^{\gamma-\varepsilon}\right),$$

and so

$$\int_5^{N^\delta} |\mathbb{E}(f(z))|\,\mathrm{d}\eta = \int_5^{N^\delta}\left(\frac{C\cdot m_{sc}^5(z)}{1 - m_{sc}^2(z)} + \mathrm{O}\left(N^{\gamma-\varepsilon}\right)\right)\mathrm{d}\eta = \mathrm{O}(1) + \mathrm{O}\left(N^{\gamma-\varepsilon+\delta}\right). \tag{A.21}$$

This completes the proof of Lemma A.1. □

Appendix B: Fluctuations of Individual Eigenvalues

In this appendix, we prove Theorem 1.6. The main observation is that the determinant corresponds to linear statistics for the function $\Re\log$, while individual eigenvalue fluctuations correspond to the central limit theorem for $\Im\log$. We build on this parallel below. The main step is Proposition B.1, which considers only the case $m = 1$, the proof for the multidimensional central limit theorem being strictly similar.

In analogy with (4.5), for any $\eta \geqslant 0$, define

$$\Im\log(E + \mathrm{i}\eta) = \Im\log(E + \mathrm{i}\infty) - \int_\eta^\infty \Re\left(\frac{1}{E - \mathrm{i}u}\right)\mathrm{d}u, \tag{B.1}$$

with the convention that $\Im\log(E + \mathrm{i}\infty) = \frac{\pi}{2}$. Then we can write

$$\Im\log(E + \mathrm{i}\eta) = \frac{\pi}{2} - \arctan\left(\frac{E}{\eta}\right), \tag{B.2}$$

and as $\eta \to 0^+$, we have

$$\Im\log(E) = \begin{cases} 0 & E > 0 \\ \pi & E < 0. \end{cases}$$

Proposition B.1. *Let W be a real Wigner matrix satisfying (1.5). Then with $\Im\log\det(W - E)$ defined as*

$$\Im\log(\det(W - E)) = \sum_{k=1}^{N}\Im\log(\lambda_k - E),$$

we have

$$\frac{\frac{1}{\pi}\Im\log(\det(W - E)) - N\int_{-\infty}^{E}\rho_{sc}(x)\,\mathrm{d}x}{\frac{1}{\pi}\sqrt{\log N}} \to \mathscr{N}(0,1). \tag{B.3}$$

If W is a complex Wigner matrix satisfying (1.5), then

$$\frac{\frac{1}{\pi}\Im\log(\det(W - E)) - N\int_{-\infty}^{E}\rho_{sc}(x)\,\mathrm{d}x}{\frac{1}{\pi}\sqrt{\frac{1}{2}\log N}} \to \mathscr{N}(0,1). \tag{B.4}$$

Before proving Proposition B.1, we prove Lemma B.2 which establishes Theorem 1.6 with $m = 1$, assuming Proposition B.1.

Lemma B.2. *Proposition B.1 and Theorem 1.6 are equivalent.*

Proof. We discuss the real case, the complex case being identical. We use the notation

$$X_k = \frac{\lambda_k - \gamma_k}{\sqrt{\frac{4\log N}{\left(4-\gamma_k^2\right)N^2}}}, \quad Y_k(\xi) = \left|\left\{ j \,:\, \lambda_j \leqslant \gamma_k + \xi\sqrt{\frac{4\log N}{\left(4-\gamma_k^2\right)N^2}} \right\}\right|,$$

with X_k as in (1.12). Let

$$e\left(Y_k(\xi)\right) = N\int_{-2}^{\gamma_k+\xi\sqrt{\frac{4\log N}{\left(4-\gamma_k^2\right)N^2}}} \rho_{sc}(x)\,\mathrm{d}x, \quad v\left(Y_k(\xi)\right) = \frac{1}{\pi}\sqrt{\log N}.$$

The main observation is that

$$\mathbb{P}\left(X_k < \xi\right) = \mathbb{P}\left(Y_k(\xi) \geqslant k\right) = \mathbb{P}\left(\frac{Y_k(\xi) - e\left(Y_k(\xi)\right)}{v\left(Y_k(\xi)\right.} \geqslant \frac{k - e\left(Y_k(\xi)\right)}{v\left(Y_k(\xi)\right.}\right).$$

Now observe that by (1.11),

$$N\int_{-2}^{\gamma_k+\xi\sqrt{\frac{4\log N}{\left(4-\gamma_k^2\right)N^2}}} \rho_{sc}(x)\,\mathrm{d}x = k + \frac{\xi}{\pi}\sqrt{\log N} + \mathrm{o}\left(1\right).$$

This proves the claimed equivalence. □

The proof of Proposition B.1 closely follows the proof of Theorem 1.2. In particular, the proof proceeeds by comparison with GOE and GUE. In the following, we first state what is known in the GOE and GUE cases. Then we indicate the modifications to the proof of Theorem 1.2 required to establish Proposition B.1.

The GOE and GUE cases. Gustavsson [31] first established the following central limit theorem in the GUE case, and O'Rourke [45] established the GOE case. Here the notation $k(N) \sim N^\theta$ is as in (1.10).

Theorem B.3 (Theorem 1.3 in [31], Theorem 5 in [45])**.** *Let $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ be the eigenvalues of a GOE (GUE) matrix. Consider $\{\lambda_{k_i}\}_{i=1}^m$ such that $0 < k_{i+1} - k_i \sim N^{\theta_i}$, $0 < \theta_i \leqslant 1$, and $k_i/N \to a_i \in (0,1)$ as $N \to \infty$. With γ_k as in (1.11), let*

$$X_i = \frac{\lambda_{k_i} - \gamma_{k_i}}{\sqrt{\frac{4\log N}{\beta\left(4-\gamma_{k_i}^2\right)N^2}}}, \quad i = 1, \ldots, m,$$

where $\beta = 1, 2$ corresponds to the GOE, GUE cases respectively. Then as $N \to \infty$,

$$\mathbb{P}\left\{X_1 \leqslant \xi_1, \ldots, X_m \leqslant \xi_m\right\} \to \Phi_\Lambda\left(\xi_1, \ldots, \xi_m\right),$$

where Φ_Λ is the cumulative distribution function for the m-dimensional normal distribution with covariance matrix $\Lambda_{i,j} = 1 - \max\left\{\theta_k : i \leqslant k < j < m\right\}$ if $i < j$, and $\Lambda_{i,i} = 1$.

By Lemma B.2, the real (complex) case in Proposition B.1 holds for the GOE (GUE) case. Therefore we can prove Proposition B.1 by comparison, presenting only what differs from the proof of Theorem 1.2. We only consider the real case, the proof in the complex case being similar. Each step below corresponds to a section in our proof of Theorem 1.2.

Step 1: Initial Regularization.

Proposition B.4. *Let $y_1 < y_2 < \cdots < y_N$ denote the eigenvalues of a Wigner matrix satisfying (1.5). Set*

$$g(\eta) = \Im\sum_k \left(\log\left(y_k + \mathrm{i}\eta\right) - \log y_k\right) - \int_0^\eta N\Re\left(m_{sc}(\mathrm{i}s)\right)\mathrm{d}s,$$

and recall $\eta_0 = \frac{e^{(\log N)^{\frac14}}}{N}$. Then $g\left(\eta_0\right)$ converges to 0 in probability as $N \to \infty$.

Proof. Again, we choose $\tilde{\eta} = \frac{c_N}{N} = \frac{(\log N)^{\frac{1}{4}}}{N}$. Then

$$\mathbb{E}\left|g\left(\eta_0\right) - g\left(\tilde{\eta}\right)\right| \leqslant \mathbb{E}\int_{\tilde{\eta}}^{\eta_0} N\left|\Re\left(s(\mathrm{i}u)\right) - \Re\left(m_{sc}\left(\mathrm{i}u\right)\right)\right| \mathrm{d}u.$$

Theorem 2.2 holds whether we consider s or $\Im(s)$, so that exactly the same argument as previously shows $\mathbb{E}\left|g\left(\eta_0\right) - g\left(\tilde{\eta}\right)\right| = \mathrm{o}\left(\sqrt{\log N}\right)$.

Next define $b_N = \frac{e^{-(\log N)^{\frac{1}{8}}}}{N}$. As b_N is below the microscopic scale, by Corollary 2.7,

$$\sum_{|x_k| \leqslant b_N} \left(\Im \log\left(x_k + \mathrm{i}\tilde{\eta}\right) - \Im \log\left(x_k\right)\right)$$

converges to 0 in probability, as the probability it is an empty sum converges to 1.

Consider now

$$\sum_{|x_k| > b_N} \left(\Im \log\left(x_k + \mathrm{i}\tilde{\eta}\right) - \Im \log\left(x_k\right)\right). \tag{B.5}$$

Let $N_1(u) = |\{x_k \leqslant u\}|$ and note that

$$\Im \log(x) - \Im \log(x + \mathrm{i}\tilde{\eta}) = \int_0^{\tilde{\eta}} \Re\left(\frac{1}{x - \mathrm{i}u}\right) \mathrm{d}u = \arctan\left(\frac{\tilde{\eta}}{x}\right).$$

To prove (B.5) is negligible, it is therefore enough to bound $\mathbb{E}(|X|)$ where

$$X = \int_{b_N \leqslant |x| \leqslant 10} \arctan\left(\frac{\tilde{\eta}}{x}\right) \mathrm{d}N_1(x) = \int_{b_N}^{10} \arctan\left(\frac{\tilde{\eta}}{x}\right) \mathrm{d}(N_1(x) + N_1(-x) - 2N_1(0)).$$

After integration by parts, the boundary terms are $\mathrm{o}(1)$ and

$$\tilde{\eta}\int_{b_N}^{10} \frac{\mathbb{E}(|N_1(x) + N_1(-x) - 2N_1(0)|)}{x^2 + \tilde{\eta}^2} \mathrm{d}x$$

remains. Split the above integral into integrals over $[b_N, a]$ and $[a, 10]$ where $a = \exp(C(\log\log N)^2)/N$ for a large enough C. On the first domain, Corollary 2.6 gives the bound $\mathbb{E}(|N_1(x)+N_1(-x)-2N_1(0)|) \leqslant CNx+\delta$ for any small $\delta > 0$. On the second domain, by rigidity [22] we have $|N_1(x) + N_1(-x) - 2N_1(0)| \leqslant \exp(C(\log\log N)^2)$, so that the contribution from this term is also $\mathrm{o}\left(\sqrt{\log N}\right)$. □

Step 2: Coupling of Determinants. With the notation of Section 3 we have,

$$e^{t/2}\Im\left(f_t\left(\mathrm{i}\eta_0\right)\right) = \frac{\mathrm{d}}{\mathrm{d}\nu}\sum_{k=1}^{N}\left(\Im \log\left(\lambda_k^{(\nu)}(t) + \mathrm{i}\eta_0\right)\right).$$

We can therefore proceed in the same way as Proposition 3.2 to prove the following.

Proposition B.5. *Let $\varepsilon > 0$, $\tau = N^{-\varepsilon}$ and let z_τ be as in (3.5) with $z = \mathrm{i}\eta_0$. Let*

$$g(t, \eta) = \sum_k \left(\Im \log\left(x_k(t) + \mathrm{i}\eta\right) - \Im \log\left(y_k(t) + \mathrm{i}\eta\right)\right)$$

Then for any $\delta > 0$, $\lim_{N\to\infty} \mathbb{P}\left(\left|g\left(\tau, \eta_0\right) - g\left(0, z_\tau\right)\right| > \delta\right) = 0$.

Step 3: Conclusion of the Proof. We reproduce the reasonning from (4.11) to (4.12) to prove Proposition B.1 in the real symmetric case. From [45] and Proposition B.4, for some explicit deterministic c_N we have

$$\frac{\sum_{k=1}^N \operatorname{Im} \log\left(x_k(\tau)+\mathrm{i}\eta_0\right)+c_N}{\sqrt{\log N}} \to \mathscr{N}(0,1), \tag{B.6}$$

and Proposition B.5 implies that

$$\frac{\sum_{k=1}^N \operatorname{Im} \log\left(y_k(\tau)+\mathrm{i}\eta_0\right)+c_N}{\sqrt{\log N}} + \frac{\sum_{k=1}^N \operatorname{Im} \log\left(x_k(0)+z_\tau\right)-\sum_{k=1}^N \operatorname{Im} \log\left(y_k(0)+z_\tau\right)}{\sqrt{\log N}} \to \mathscr{N}(0,1).$$

Lemmas B.6 and B.7 show that the second term above, call it X, satisfies $\mathbb{E}(X^2)<C\varepsilon$, for some universal C. Thus for any fixed smooth and compactly supported function F,

$$\begin{aligned}\mathbb{E}\left(F\left(\frac{\sum_{k=1}^N \operatorname{Im} \log\left(y_k(\tau)+\mathrm{i}\eta_0\right)+c_N}{\sqrt{\log N}}\right)\right) &= \mathbb{E}\left(F\left(\frac{\sum_{k=1}^N \operatorname{Im} \log\left(x_k(\tau)+\mathrm{i}\eta_0\right)|+c_N}{\sqrt{\log N}}+X\right)\right)+\mathrm{O}\left(\|F\|_{\mathrm{Lip}}(\mathbb{E}\left(X^2\right))^{1/2}\right)\\ &= \mathbb{E}\left(F(\mathscr{N}(0,1))\right)+\mathrm{o}(1)+\mathrm{O}\left(\varepsilon^{1/2}\right).\end{aligned}$$

With Theorem 4.4 (its proof applies equally to the imaginary part), the above equation implies

$$\mathbb{E}\left(F\left(\frac{\operatorname{Im}\log\det(W+\mathrm{i}\eta_0)+c_N}{\sqrt{\log N}}\right)\right)=\mathbb{E}\left(F(\mathscr{N}(0,1))\right)+\mathrm{o}(1)+\mathrm{O}\left(\varepsilon^{1/2}\right),$$

and by Proposition B.4, we obtain

$$\mathbb{E}\left(F\left(\operatorname{Im}\frac{\log\det W+\frac{N}{2}}{\sqrt{\log N}}\right)\right)=\mathbb{E}\left(F(\mathscr{N}(0,1))\right)+\mathrm{o}(1)+\mathrm{O}\left(\varepsilon^{1/2}\right). \tag{B.7}$$

Since ε is arbitrarily small, this concludes the proof.

Lemma B.6. *Recall the notation $\tau=N^{-\varepsilon}$ and let $\{x_k\}_{k=1}^N$, $\{y_k\}_{k=1}^N$ denote the eigenvalues of two Wigner matrices, W_1 and W_2. Then*

$$\lim_{N\to\infty}\mathbb{E}\left(\sum_{k=1}^N \Im\log\left(x_k+\mathrm{i}\tau\right)-\sum_{k=1}^N \Im\log\left(y_k+\mathrm{i}\tau\right)\right)=\mathrm{O}(1).$$

The proof of this lemma requires only trivial adjustments of the proof of Lemma A.1, details are left to the reader. Finally, we also have the following bound on the variance.

Lemma B.7. *Recall the notation $\tau=N^{-\varepsilon}$ and let $\{x_k\}_{k=1}^N$, denote the eigenvalues of a Wigner matrix W. Then there exists $\varepsilon_0>0$ such that for any $0<\varepsilon<\varepsilon_0$ we have*

$$\operatorname{Var}\left(\sum_{k=1}^N \Im\log\left(x_k+\mathrm{i}\tau\right)\right)\leqslant C(1+\varepsilon\log N). \tag{B.8}$$

For the proof, let $\chi_{[-5,5]}$ is a smooth indicator of the interval $[-5,5]$ and $\varphi_N(x)=\chi(x)\Im\log\left(x+\mathrm{i}\tau\right)$. Our first proof of Proposition 4.5 shows it is enough to check that $\int|\hat\varphi_N(\xi)|^2|\xi|\,\mathrm{d}\xi=\mathrm{O}\left(1+\log\tau\right)$. We can verify this bound by integrating by parts as before. Alternatively, we can use the second proof of Proposition 4.5 based on the resolvent, which applies without changes.

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