Séminaire Poincaré XIV (2010) 115 – 153

Quantum chaos, random matrix theory, and the Riemann ζ-function

Paul Bourgade
Télécom ParisTech
23, Avenue d'Italie
75013 Paris, FR.

Jonathan P. Keating
University of Bristol
University Walk, Clifton
Bristol BS8 1TW, UK.

Hilbert and Pólya put forward the idea that the zeros of the Riemann zeta function may have a spectral origin : the values of t_n such that $\frac{1}{2} + \mathrm{i}t_n$ is a non trivial zero of ζ might be the eigenvalues of a self-adjoint operator. This would imply the Riemann Hypothesis. From the perspective of Physics one might go further and consider the possibility that the operator in question corresponds to the quantization of a classical dynamical system.

The first significant evidence in support of this spectral interpretation of the Riemann zeros emerged in the 1950's in the form of the resemblance between the Selberg trace formula, which relates the eigenvalues of the Laplacian and the closed geodesics of a Riemann surface, and the Weil explicit formula in number theory, which relates the Riemann zeros to the primes. More generally, the Weil explicit formula resembles very closely a general class of Trace Formulae, written down by Gutzwiller, that relate quantum energy levels to classical periodic orbits in chaotic Hamiltonian systems.

The second significant evidence followed from Montgomery's calculation of the pair correlation of the t_n's (1972) : the zeros exhibit the same repulsion as the eigenvalues of typical large unitary matrices, as noted by Dyson. Montgomery conjectured more general analogies with these random matrices, which were confirmed by Odlyzko's numerical experiments in the 80's.

Later conjectures relating the statistical distribution of random matrix eigenvalues to that of the quantum energy levels of classically chaotic systems connect these two themes.

We here review these ideas and recent related developments : at the rigorous level strikingly similar results can be independently derived concerning number-theoretic L-functions and random operators, and heuristics allow further steps in the analogy. For example, the t_n's display Random Matrix Theory statistics in the limit as $n \to \infty$, while lower order terms describing the approach to the limit are described by non-universal (arithmetic) formulae similar to ones that relate to semiclassical quantum eigenvalues. In another direction and scale, macroscopic quantities, such as the moments of the Riemann zeta function along the critical line on which the Riemann Hypothesis places the non-trivial zeros, are also connected with random matrix theory.

1 First steps in the analogy

This section describes the fundamental mathematical concepts (i.e. the Riemann zeta function and random operators) the connections between which are the focus of this survey : linear statistics (trace formulas) and microscopic interactions (fermionic repulsion). These statistical connections have since been extended to many other L-functions (in the Selberg class [45], over function fields [39]) : for the sake of brevity we only consider the Riemann zeta function.

1.1 Basic theory of the Riemann zeta function

The Riemann zeta function can be defined, for $\sigma = \Re(s) > 1$, as a Dirichlet series or an Euler product :

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \mathcal{P}} \frac{1}{1 - \frac{1}{p^s}},$$

where $\mathcal{P}$ is the set of all prime numbers. The second equality is a consequence of the unique factorization of integers into prime numbers. A remarkable fact about this function, proved in Riemann's original paper, is that it can be meromorphically extended to the complex plane, and that this extension satisfies a functional equation.

Theorem 1.1. *The function ζ admits an analytic extension to $\mathbb{C} - \{1\}$ which satisfies the equation (writing $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$)*

$$\xi(s) = \xi(1 - s).$$

Proof. The gamma function is defined for $\Re(s) > 0$ by $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} \mathrm{d}t$, hence, substituting $t = \pi n^2 x$,

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \frac{1}{n^s} = \int_0^\infty x^{\frac{s}{2}-1} e^{-n^2 \pi x} \mathrm{d}x.$$

If we sum over n, the sum and integral can be exchanged for $\Re(s) > 1$ because of absolute convergence, hence

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^\infty x^{\frac{s}{2}-1} \omega(x) \mathrm{d}x \tag{1}$$

for $\omega(x) = \sum_{n=1}^\infty e^{-n^2 \pi x}$. The Poisson summation formula implies that the Jacobi theta function $\theta(x) = \sum_{n=-\infty}^\infty e^{-n^2 \pi x}$ satisfies the functional equation

$$\theta\left(\frac{1}{x}\right) = \sqrt{x}\, \theta(x),$$

hence $2\omega(x) + 1 = (2\omega(1/x) + 1)/\sqrt{x}$. Equation (1) therefore yields, by first splitting the integral at $x = 1$ and substituting $1/x$ for x between 0 and 1,

$$\begin{aligned}
\xi(s) &= \int_1^\infty x^{\frac{s}{2}-1} \omega(x) \mathrm{d}x + \int_1^\infty x^{-\frac{s}{2}-1} \omega\left(\frac{1}{x}\right) \mathrm{d}x \\
&= \int_1^\infty x^{\frac{s}{2}-1} \omega(x) \mathrm{d}x + \int_1^\infty x^{-\frac{s}{2}-1} \left(-\frac{1}{2} + \frac{\sqrt{x}}{2} + \sqrt{x}\omega\left(x\right)\right) \mathrm{d}x \\
\xi(s) &= \frac{1}{s(s-1)} + \int_1^\infty \left(x^{-\frac{s}{2}-\frac{1}{2}} + x^{\frac{s}{2}-1}\right) \omega(x) \mathrm{d}x,
\end{aligned}$$

still for $\Re(s) > 1$. The right hand side is properly defined on $\mathbb{C} - \{0, 1\}$ (because $\omega(x) = \mathrm{O}(e^{-\pi x})$ as $x \to \infty$) and invariant under the substitution $s \to 1 - s$: the expected result follows. $\square$

From the above theorem, the zeta function admits trivial zeros at $s = -2, -4, -6, \dots$ corresponding to the poles of $\Gamma(s/2)$. All all non-trivial zeros are confined in the critical strip $0 \leq \sigma \leq 1$, and they are symmetrically positioned about the real axis and the critical line $\sigma = 1/2$. The Riemann hypothesis asserts that they all lie on this line.

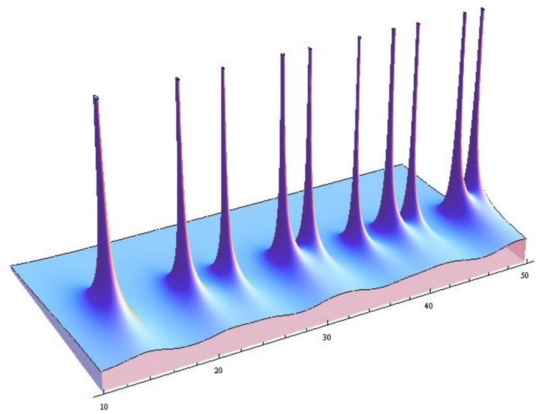


Fig. 1 – The first ζ zeros : $1/|\zeta|$ in the domain $-2 < \sigma < 2,\ 10 < t < 50$.

One can define the argument of $\zeta(s)$ continuously along the line segments from 2 to 2+it to 1/2+it. Then the number of such zeros ρ counted with multiplicities in $0 < \Im(\rho) < t$ is asymptotically (as shown by a calculus of residues)

$$\mathcal{N}(t) = \frac{t}{2\pi} \log \frac{t}{2\pi e} + \frac{1}{\pi} \arg \zeta \left(\frac{1}{2} + \mathrm{i}t \right) + \frac{7}{8} + \mathrm{O}\left(\frac{1}{t} \right). \tag{2}$$

In particular, the mean spacing between ζ zeros at height t is is $2\pi / \log |t|$.

The fact that there are no zeros on $\sigma = 1$ led to the proof of the prime number theorem, which states that

$$\pi(x) \underset{x\to\infty}{\sim} \frac{x}{\log x}, \tag{3}$$

where $\pi(x) = |\mathcal{P} \cap [\![1, x]\!]|$. The proof makes use of the Van Mangoldt function, $\Lambda(n) = \log p$ if n is a power of a prime p, 0 otherwise : writing $\psi(x) = \sum_{n \leq x} \Lambda(n)$, (3) is equivalent to $\lim_{x\to\infty} \Psi(x)/x = 1$, because obviously $\psi(x) \leq \pi(x) \log x$ and, for any $\varepsilon > 0$, $\psi(x) \geq \sum_{x^{1-\varepsilon} \leq p \leq x} \log p \geq (1 - \varepsilon)(\log x)(\pi(x) + \mathrm{O}(x^{1-\varepsilon}))$. Differentiating the Euler product for ζ, if $\Re(s) > 1$,

$$-\frac{\zeta'}{\zeta}(s) = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s},$$

which allows one to transfer the problem of the asymptotics of ψ to analytic properties of ζ : for $c > 0$, by a residues argument

$$\frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{y^s}{s} \mathrm{d}s = 0 \text{ if } 0 < y < 1,\ 1 \text{ if } y > 1,$$

hence

$$\begin{aligned} \psi(x) = \sum_{n=2}^{\infty} \Lambda(n) \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{(x/n)^s}{s} \mathrm{d}s \\ = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}t}^{c+\mathrm{i}t} \left(-\frac{\zeta'}{\zeta}(s) \right) \mathrm{d}s \frac{x^s}{s} \mathrm{d}s + \mathrm{O}\left(\frac{x \log^2 x}{t} \right), \end{aligned} \tag{4}$$

where the error term, created by the bounds restriction, is made explicit and small by the choice $c = 1 + 1/\log x$. Assuming the Riemann hypothesis for the moment, for any $1/2 < \sigma < 1$, one can change the integral path from $c - \mathrm{i}t, c + it$ to $\sigma + \mathrm{i}t, \sigma - \mathrm{i}t$ by just crossing the pole at $s = 1$, with residue x :

$$\psi(x) = x + \frac{1}{2\pi \mathrm{i}} \int_{\sigma - \mathrm{i}t}^{\sigma + \mathrm{i}t} \left(-\frac{\zeta'}{\zeta}(s) \right) \mathrm{d}s \frac{x^s}{s} \mathrm{d}s + \mathrm{O}\left(\frac{x \log^2 x}{t} \right).$$

Independently, still under the Riemann hypothesis, one can show the bound $\zeta'/\zeta(\sigma + \mathrm{i}t) = \mathrm{O}(\log t)$, implying $\psi(x) = x + \mathrm{O}(x^\theta)$ for any $\theta > 1/2$, by choosing $t = x$. What if we do not assume the Riemann hypothesis ? The above reasoning can be reproduced, giving a worse error bound, provided that the integration path on the right hand side of (4) can be changed crossing only one pole and making ζ'/ζ small by approaching sufficiently the critical axis. This is essentially what was proved independently in 1896 by Hadamard and La Vallée Poussin, who showed that $\zeta(\sigma + \mathrm{i}t)$ cannot be zero for $\sigma > 1 - c/\log t$, for some $c > 0$. This finally yields

$$\pi(x) = \mathrm{Li}(x) + \mathrm{O}\left(x e^{-c\sqrt{\log x}} \right), \text{ where } \mathrm{Li}(x) = \int_0^x \frac{\mathrm{d}s}{\log s},$$

while the Riemann hypothesis would imply $\pi(x) = \mathrm{Li}(x) + \mathrm{O}(\sqrt{x} \log x)$. It would also have consequences for the extreme size of the zeta function (the Lindelöf hypothesis) : for any $\varepsilon > 0$

$$\zeta(1/2 + \mathrm{i}t) = \mathrm{O}(t^\varepsilon),$$

which is equivalent to bounds on moments of ζ discussed in Section 3.

1.2 The explicit formula

We now consider the first analogy between the zeta zeros and spectral properties of operators, by looking at linear statistics. Namely, we state and give key ideas underlying the proofs of Weil's explicit formula concerning the ζ zeros and Selberg's trace formula for the Laplacian on surfaces with constant negative curvature.

First consider the Riemann zeta function. For a function $f : (0, \infty) \to \mathbb{C}$ consider its Mellin transform $F(s) = \int_0^\infty f(s) x^{s-1} \mathrm{d}x$. Then the inversion formula (where σ is chosen in the fundamental strip, i.e. where the image function F converges)

$$f(x) = \frac{1}{2\pi \mathrm{i}} \int_{\sigma - \mathrm{i}\infty}^{\sigma + \mathrm{i}\infty} F(s) x^{-s} \mathrm{d}s$$

holds under suitable smoothness assumptions, in a similar way as the inverse Fourier transform. Hence, for example,

$$\sum_{n=2}^{\infty} \Lambda(n) f(n) = \sum_{n=2}^{\infty} \Lambda(n) \frac{1}{2\pi \mathrm{i}} \int_{2 - \mathrm{i}\infty}^{2 + \mathrm{i}\infty} F(s) n^{-s} \mathrm{d}s = \frac{1}{2\pi \mathrm{i}} \int_{2 - \mathrm{i}\infty}^{2 + \mathrm{i}\infty} \left(-\frac{\zeta'}{\zeta} \right) (s) F(s) \mathrm{d}s.$$

Changing the line of integration from $\Re(s) = 2$ to $\Re(s) = -\infty$, all trivial and non-trivial poles (as well as $s = 1$) are crossed, leading to the following explicit formula by Weil.

Theorem 1.2. *Suppose that f is $\mathscr{C}^2$ on $(0,\infty)$ and compactly supported. Then*

$$\sum_\rho F(\rho)+\sum_{n\geq 0}F(-2n)=F(1)+\sum_{p\in\mathcal{P},m\in\mathbb{N}}(\log p)\,f(p^m),$$

where the first sum the first sum is over non-trivial zeros counted with multiplicities.

When replacing the Mellin transform by the Fourier transform, the Weil explicit formula takes the following form, for an even function h, analytic on $|\Im(z)|<1/2+\delta$, bounded, and decreasing as $h(z)=\mathrm{O}(|z|^{-2-\delta})$ for some $\delta>0$. Here, the sum is over all γ_n's such that $1/2+\mathrm{i}\gamma_n$ is a non-trivial zero, and $\hat h(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}h(y)e^{-\mathrm{i}xy}\mathrm{d}y$:

$$\sum_{\gamma_n}h(\gamma_n)-2h\left(\frac{i}{2}\right)=\frac{1}{2\pi}\int_{\mathbb{R}}h(r)\left(\frac{\Gamma'}{\Gamma}\left(\frac14+\frac{i}{2}r\right)-\log\pi\right)\mathrm{d}r$$
$$-2\sum_{p\in\mathcal{P}m\in\mathbb{N}}\frac{\log p}{p^{m/2}}\hat h(m\log p). \quad (5)$$

A formally similar relation holds in a different context, through Selberg's trace formula. In one of its simplest manifestations, it can be stated as follows. Let $\Gamma\backslash\mathbb{H}$ be an hyperbolic surface, where Γ is a subgroup of $\mathrm{PSL}_2(\mathbb{R})$, orientation-preserving isometries of the hyperbolic plane $\mathbb{H}=\{x+\mathrm{i}y,y>0\}$, the Poincaré half-plane with metric

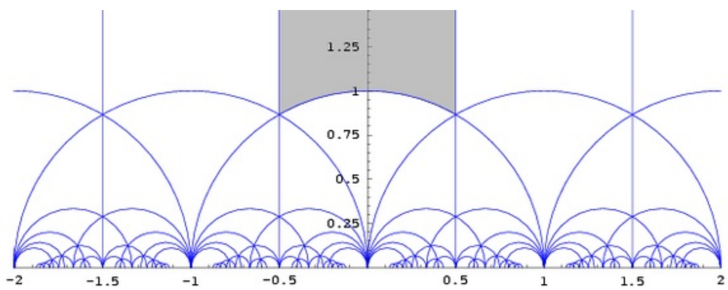


FIG. 2 – Geodesics and a fundamental domain (for the modular group) in the hyperbolic plane.

$$\mathrm{d}\mu=\frac{\mathrm{d}x\mathrm{d}y}{y^2}. \quad (6)$$

The Laplace-Beltrami operator $\Delta=y^2(\partial_{xx}+\partial_{yy})$ is self-adjoint with respect to the invariant measure (6), i.e. $\int v(\Delta u)\mathrm{d}\mu=\int(\Delta v)u\mathrm{d}\mu$, so all eigenvalues of Δ are real and positive. If $\Gamma\backslash\mathbb{H}$ is compact, the spectrum of Δ restricted to a fundamental domain $\mathcal{D}$ of representatives of the conjugation classes is discrete, $0=\lambda_0<\lambda_1<\dots$ with associated eigenfunctions $u_1,u_2,\dots$:

$$\left\{\begin{array}{rcl}(\Delta+\lambda_n)u_n&=&0,\\ u_n(\gamma z)&=&u_n(z)\text{ for all }\gamma\in\Gamma,\,z\in\mathbb{H}.\end{array}\right.$$

To state Selberg's trace formula, we need, as previously, a function h analytic on $|\Im(z)|<1/2+\delta$, even, bounded, and decreasing as $h(z)=\mathrm{O}(|z|^{-2-\delta})$, for some $\delta>0$.

Theorem 1.3. *Under the above hypotheses, setting $\lambda_k=s_k(1-s_k)$, $s_k=1/2+\mathrm{i}r_k$, then*

$$\sum_{k=0}^{\infty}h(r_k)=\frac{\mu(\mathcal{D})}{2\pi}\int_{-\infty}^{\infty}rh(r)\tanh(\pi r)\mathrm{d}r+\sum_{p\in\mathcal{P},m\in\mathbb{N}^*}\frac{\ell(p)}{2\sinh\left(\frac{m\ell(p)}{2}\right)}\hat h(m\ell(p)), \quad (7)$$

where $\hat{h}$ is the Fourier transform of h ($\hat{h}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(y)e^{-\mathrm{i}xy}\mathrm{d}y$), $\mathcal{P}$ is now the set of all primitive[1] *periodic orbits*[2] *and ℓ is the geodesic distance for the metric (6).*

Sketch of proof. It is a general fact that the eigenvalue density function $d(\lambda)$ is linked to the Green function associated to λ ($(\Delta+\lambda)G^{(\lambda)}(z,z') = \delta_{z-z'}$, where δ is the Dirac distribution at 0) through

$$d(\lambda) = -\frac{1}{\pi}\int_{\mathcal{D}} \Im\left(G^{(\lambda)}(z,z)\right)\mathrm{d}\mu.$$

To calculate $G^{(\lambda)}$, we need to sum the Green function associated to the whole Poincaré half plane over the images of z by elements of Γ (in the same way as the transition probability from z' to z is the sum of all transition probabilities to images of z) :

$$G^{(\lambda)}(z,z') = \sum_{\gamma\in\Gamma} G^{(\lambda)}_{\mathbb{H}}(\gamma(z),z').$$

Thanks to the numerous isometries of $\mathbb{H}$, the geodesic distance for the Poincaré plane is well-known. This yields an explicit form of the Green function, leading to

$$d(\lambda) = \frac{1}{2\sqrt{2}\pi^2}\sum_{\gamma\in\Gamma}\int \mathrm{d}\mu(z)\int_{\ell(z,\gamma(z))}^{\infty} \frac{\sin(rs)}{\sqrt{\cosh s - \cosh \ell(z,\gamma(z))}}\mathrm{d}s,$$

with $\lambda = 1/4 + r^2$. The mean density of states corresponds to $\gamma = \mathrm{Id}$ and an explicit calculation yields

$$\langle d(\lambda)\rangle = \frac{\mu(\mathcal{D})}{4\pi}\tanh(\pi r).$$

It is not clear at this point how the primitive periodic orbits appear from the elements in Γ. The sum over group elements γ can be written as a sum over conjugacy classes $\overline{\gamma}$. This gives

$$d(\lambda) = \langle d(\lambda)\rangle + \sum_{\overline{\gamma}} d_{\overline{\gamma}}(\lambda)$$

where

$$d_{\overline{\gamma}}(\lambda) = \frac{1}{2\sqrt{2}\pi^2}\int_{\mathrm{FD}(\overline{\gamma})} \mathrm{d}\mu(z)\int_{\ell(z,\gamma(z))}^{\infty} \frac{\sin(rs)}{\sqrt{\cosh s - \cosh \ell(z,\gamma(z))}}\mathrm{d}s,$$

where $\mathrm{FD}(\overline{\gamma})$ is the fundamental domain associated to the subgroup S_γ of elements commuting with γ (independent of the representant of the conjugacy class). The subgroup S_γ is generated by an element γ_0 :

$$S_\gamma = \{\gamma_0^m, m \in \mathbb{Z}\}.$$

Then an explicit (but somewhat tedious) calculation gives

$$d_{\overline{\gamma}}(\lambda) = \frac{\ell^{(0)}(p)}{4\pi\sinh\left(\frac{s\ell(p)}{2}\right)}\cos(s\ell(p)),$$

[1] i.e. not the repetition of shorter periodic orbits
[2] of the geodesic flow on $\Gamma\backslash\mathbb{H}$

where $\ell^{(0)}(p)$ (resp. $\ell(p)$) is defined by $2\cosh \ell^{(0)}(p) = \mathrm{Tr}(\gamma_0)$ (resp. $2\cosh \ell(p) = \mathrm{Tr}(\gamma)$). Independent calculation shows that $\ell^{(0)}(p)$ (resp. $\ell(p)$) is also the length between z and $\gamma_0(z)$ (resp. z and $\gamma(z)$). Hence they are the lengths of the unique (up to conjugation) periodic orbits associated to γ_0 (resp. γ). The above proof sketch can be made rigorous by integrating d with respect to a suitable test function h. $\square$

The similarity between both explicit formulas (5) and (7) suggests that prime numbers may correspond to primitive orbits, with lengths $\log p, p \in \mathcal{P}$. This analogy remains when counting primes and primitive orbits. Indeed, as a consequence of Selberg's trace formula, the number of primitive orbits with length less than x is

$$|\{\ell(p) < x\}| \underset{x\to\infty}{\sim} \frac{e^x}{x},$$

and following the prime number theorem (3),

$$|\{\log(p) < x\}| \underset{x\to\infty}{\sim} \frac{e^x}{x}.$$

These connections are reviewed at much greater length in [5].

Finally, note that the signs of the oscillating parts are different between equation (5) and (7). One explanation by Connes [12] suggests that the ζ zeros may not be in the spectrum of an operator but in its absorption : for an Hermitian operator with continuous spectrum along the whole real axis, they would be exactly the missing points where the eigenfunctions vanish.

1.3 Basic theory of random matrices eigenvalues

As we will see in the next section, the correlations between ζ zeros show striking similarities with those known to exist between the eigenvalues of random matrices. This adds further weight to the idea that there may be a spectral interpretation of the zeros and provides another link with the theory of quantum chaotic systems. We need first to introduce the matrices we will consider. These have the property that their spectrum has an explicit joint distribution, exhibiting a two-point repulsive interaction, like fermions. Importantly, these correlations have a *determinantal* structure.

If $\chi = \sum_i \delta_{X_i}$ is a simple point process on a complete separate metric space Λ, consider the point process

$$\Xi^{(k)} = \sum_{X_{i_1},\dots,X_{i_k}\text{all distinct}} \delta_{(X_{i_1},\dots,X_{i_k})} \tag{8}$$

on Λ^k. One can define in this way a measure M_k on Λ^k by $M^{(k)}(\mathcal{A}) = \mathbb{E}\left(\Xi^{(k)}(\mathcal{A})\right)$ for any Borel set $\mathcal{A}$ in Λ^k. Most of the time, there is a natural measure λ on Λ, in our cases $\Lambda = \mathbb{R}$ or $(0, 2\pi)$ and λ is the Lebesgue measure. If $M^{(k)}$ is absolutely continuous with respect to λ^k, there exists a function ρ_k on Λ^k such that for any Borel sets $B_1, \dots, B_k$ in Λ

$$M^{(k)}(B_1 \times \cdots \times B_k) = \int_{B_1\times\cdots\times B_k} \rho_k(x_1, \dots, x_k)\mathrm{d}\lambda(x_1)\dots\mathrm{d}\lambda(x_k).$$

Hence one can think about $\rho_k(x_1,\dots,x_k)$ as the asymptotic (normalized) probability of having exactly one particle in neighborhoods of the x_k's. More precisely under suitable smoothness assumptions, and for distinct points $x_1,\dots,x_k$ in $\Lambda=\mathbb{R}$,

$$\rho_k(x_1,\dots,x_k)=\lim_{\varepsilon\to 0}\frac{\mathbb{P}\left(\chi(x_i,x_i+\varepsilon)=1,1\leq i\leq k\right)}{\prod_{j=1}^k\lambda(x_j,x_j+\varepsilon)}.$$

This is called the kth-order correlation function of the point process. Note that ρ_k is not a probability density. If χ consists almost surely of n points, it satisfies the integration property

$$(n-k)\rho_k(x_1,\dots,x_k)=\int_\Lambda\rho_{k+1}(x_1,\dots,x_{k+1})\mathrm{d}\lambda(x_{k+1}). \tag{9}$$

A particularly interesting class of point processes is the following.

Definition 1.4. *If there exists a function $K:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that for all $k\geq 1$ and $(z_1,\dots,z_k)\in\mathbb{R}^k$*

$$\rho_k(z_1,\dots,z_k)=\det\left(K(z_i,z_j)_{i,j=1}^k\right)$$

then χ is said to be a determinantal point process with respect to the underlying mesure λ and correlation kernel K.

The determinantal condition for *all* correlation functions looks very restrictive, but it is not : for example, if the joint density of all n particles can be written as a Vandermonde-type determinant, then so can the lower order correlation functions, as shown by the following argument, standard in Random Matrix Theory. It shows that for a Coulomb gas at a specific temperature (1/2) in dimension 1 or 2, all correlations functions are explicit, a very noteworthy feature.

Proposition 1.5. *Let $\mathrm{d}\lambda$ be a probability measure on $\mathbb{C}$ (eventually concentrated on a line) such that for the Hermitian product*

$$(f,g)\mapsto\langle f,g\rangle=\int f\overline{g}\mathrm{d}\lambda$$

polynomial moments are defined till order at least $n-1$. Consider the probability distribution with density

$$F(x_1,\dots,x_n)=c(n)\prod_{k<l}|x_l-x_k|^2$$

with respect to $\prod_{j=1}^n\mathrm{d}\lambda(x_j)$, where $c(n)$ is the normalization constant. For this joint distribution, $\{x_1,\dots,x_n\}$ is a determinantal point process with explicit kernel.

Proof. Let P_k $(0\leq k\leq n-1)$ be monic polynomials with degree k. Thanks to Vandermonde's formula and the multilinearity of the determinant

$$\prod_{k<l}(x_l-x_k)=\sqrt[n]{\prod_{k=0}^{n-1}\|P_k\|_{\mathrm{L}^2(\lambda)}}\det\left(\frac{P_k(x_j)}{\|P_k\|_{\mathrm{L}^2(\lambda)}}\right)_{k,j=1}^n.$$

Multiplying this identity with itself and using $\det(AB)=\det A\det B$ gives

$$F(x_1,\dots,x_n)=\det\left(K_n(x_j,x_k)_{j,k=1}^n\right)$$

with $K_n(x,y) = c\sum_{k=0}^{n-1} \frac{P_k(x)\overline{P_k(y)}}{\|P_k\|^2_{\mathrm{L}^2(\lambda)}}$, the constant c depending on λ, n and the P_i's. This shows that the correlation ρ_n has the desired determinantal form. The following lemma by Gaudin (see [33]) together with the integration property (9) shows that if the polynomials P_k's are orthogonal in $\mathrm{L}^2(\lambda)$, then

$$\rho_l(x_1,\dots,x_l) = \det\left(K_n(x_j,x_k)_{j,k=1}^l\right)$$

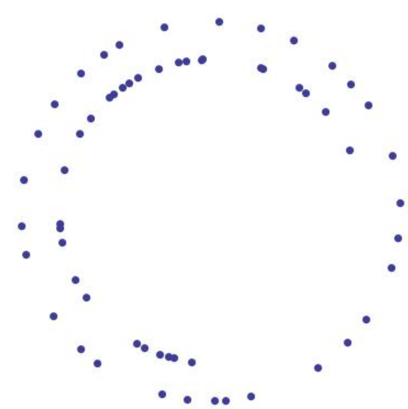
for all $1 \leq l \leq n$. The probability density condition $\int_{\mathbb{R}} \rho_1(x)\mathrm{d}\lambda(x) = n$ implies $c = 1$, and finally the Christoffel-Darboux formula for orthogonal polynomials gives

$$K_n(x,y) = \sum_{k=0}^{n-1} \frac{P_k(x)P_k(y)}{\|P_k\|^2_{\mathrm{L}^2(f)}} = \frac{1}{\|P_n\|^2_{\mathrm{L}^2(f)}}\,\frac{P_n(x)\overline{P_{n-1}(y)} - P_{n-1}(x)P_n(y)}{x-y}\,. \tag{10}$$

which concludes the proof. □

Lemma 1.6. *Suppose that the function K satisfies, for some measurable set I, the semigroup relation $\int_I K(x,y)K(y,z)\mathrm{d}\lambda(y) = K(x,z)$ for all x and z in I, and denote $n = \int_I K(x,x)\mathrm{d}\lambda(x)$. Then for all k,*

$$\int \det_{(k+1)\times(k+1)} K(x_i,x_j)\mathrm{d}\lambda(x_{k+1}) = (n-k)\det_{k\times k} K(x_i,x_j).$$



We apply the above discussion to the following examples, which are among the most studied random matrices. The first involves Hermitian matrices with Gaussian entries ; the second relates to a compact group : uniformly distributed unitary matrices. Their spectrum is a determinantal point process, which implies a *repulsion* between the eigenvalues, similar to that between fermions. We illustrate this here with one example of determinantal statistics (eigenvalues of a Haar-distributed unitary matrix, outer circle) together with, for comparison, Poisson distributed points (uniform independent points, inner circle), in dimension $n = 30$.

First, consider the so-called Gaussian unitary ensemble (GUE). This is the ensemble of random $n \times n$ Hermitian matrices with independent (up to symmetry) Gaussian entries : $M_{ij}^{(n)} = \overline{M_{ji}^{(n)}} = \frac{1}{\sqrt{n}}(X_{ij} + \mathrm{i}Y_{ij}), 1 \leq i < j \leq n$, where the X_{ij}'s and Y_{ij}'s are independent centered real Gaussians entries with mean 0 and variance $1/2$ and $M_{ii}^{(n)} = X_{ii}/\sqrt{n}$ with X_{ii} real centered Gaussians with variance 1, still independent. For this ensemble, the distribution of the eigenvalues has an explicit density

$$\frac{1}{Z_n} e^{-n\sum_{i=1}^n \lambda_i^2/2} \prod_{1\leq i<j\leq n} |\lambda_i - \lambda_j|^2 \tag{11}$$

with respect to the Lebesgue measure. We denote by (h_n) the Hermite polynomials, more precisely the successive monic polynomials orthogonal with respect to the

Gaussian weight $e^{-x^2/2}\mathrm{d}x$, and the associated normalized functions

$$\psi_k(x) = \frac{e^{-x^2/4}}{\sqrt{\sqrt{2\pi}k!}}\, h_k(x).$$

Then from (10), the set of point $\{\lambda_1, \dots, \lambda_n\}$ with law (11) is a determinantal point process with kernel (with respect to the Lebesgue measure on $\mathbb{R}$) given by

$$K^{\mathrm{GUE}(n)}(x,y) = n\frac{\psi_n(x\sqrt{n})\psi_{n-1}(y\sqrt{n}) - \psi_{n-1}(x\sqrt{n})\psi_n(y\sqrt{n})}{x-y},$$

defined by continuity when $x = y$. The Plancherel-Rotach asymptotics for the Hermite polynomials implies that, as $n \to \infty$, $K^{\mathrm{GUE}(n)}(x,x)/n$ has a non trivial limit. More precisely, the empirical spectral distribution $\frac{1}{n}\sum \delta_{\lambda_i}$ converges in probability to the semicircle law (see e.g. [1]) with density

$$\rho_{sc}(x) = \frac{1}{2\pi}\sqrt{(4-x^2)_+}$$

with respect to the Lebesgue measure. This is the asymptotic behavior of the spectrum in the macroscopic regime. The microscopic interactions between eigenvalues also can be evaluated thanks to asymptotics of the Hermite orthogonal polynomials : for any $x \in (-2,2)$, $u \in \mathbb{R}$,

$$\frac{1}{n\rho_{sc}(x)}K^{\mathrm{GUE}(n)}\left(x, x + \frac{u}{n\rho_{sc}(x)}\right) \underset{n\to\infty}{\longrightarrow} K(u) = \frac{\sin(\pi u)}{\pi u}, \tag{12}$$

leading to a repulsive correlation structure for the eigenvalues at the scale of the average gap : for example the two-point correlation function asymptotics are

$$\left(\frac{1}{n\rho_{sc}(x)}\right)^2 \rho_2^{\mathrm{GUE}(n)}\left(x, x + \frac{u}{n\rho_{sc}(x)}\right) \underset{n\to\infty}{\longrightarrow} r_2(u) = 1 - \left(\frac{\sin(\pi u)}{\pi u}\right)^2, \tag{13}$$

which vanishes at $u = 0$, while for independent points the asymptotic two points correlation function would be identically 1. A remarkable fact about the above sine kernel is that it appears universally in the limiting correlation functions of random Hermitian matrices with independent (up to symmetry) entries (the so-called Wigner ensemble). This was proved under very weak conditions on the entries in independent and complementary works by Tao-Vu and Erdös-Schlein-Yau & al (see [21]). In their result, a Wigner matrix is like a matrix from the GUE from the point of view of the variance normalization but with no Gaussianity condition. We just assume that the entries X_{ij}'s and Y_{ij}'s have a subexponential decay : for some constants c and c',

$$\mathbb{P}(|X_{ij}| \leq t^c) \leq e^{-t},\ \mathbb{P}(|Y_{ij}| \leq t^c) \leq e^{-t},\ t > c'.$$

Theorem 1.7. *Under the above hypothesis, denoting by $\rho_k^{\mathrm{Wig}(n)}$ the correlation functions associated with the eigenvalues of the Wigner matrix, for any u, ε such that $[u-\varepsilon, u+\varepsilon] \subset (-2,2)$, and any continuous compactly supported $f : \mathbb{R}^k \to \mathbb{R}$*

$$\frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon}\int_{\mathbb{R}^k}\frac{f(u_1,\dots,u_k)}{\rho_{sc}(x')^k}\rho_k^{\mathrm{Wig}(n)}\left(x' + \frac{u_1}{n\rho_{sc}(x')}, \dots, x' + \frac{u_k}{n\rho_{sc}(x')}\right)\mathrm{d}u_1\dots\mathrm{d}u_k\mathrm{d}x'$$

converges as $n \to \infty$ to (K is the above mentioned sine kernel)

$$\int_{\mathbb{R}^k} f(u_1, \dots, u_k) \det_{k \times k} (K(u_i - u_j)) \, \mathrm{d}u_1 \dots \mathrm{d}u_k.$$

Note that both works leading to the above result, through very different, proceed by comparison with the explicit GUE asymptotics.

The second classical example of a matrix-related determinantal point process is that of the eigenvalues of uniformly distributed unitary matrices. For $u_n \sim \mu_{\mathrm{U}(n)}$ ($\mu_{\mathrm{U}(n)}$ is the Haar measure[3] on the unitary group) the density of the eigenangles $0 \leq \theta_1 < \cdots < \theta_n < 2\pi$, with respect to the Lebesgue measure on the corresponding simplex is

$$\frac{1}{(2\pi)^n} \prod_{j<k} |e^{\mathrm{i}\theta_j} - e^{\mathrm{i}\theta_k}|^2.$$

In this case, the polynomials in the recipe from Proposition 1.5 are those orthogonal with respect to the Hermitian product on the unit circle $\int_{\mathscr{C}} \overline{p} q \mathrm{d}z$. These are the monomials (X^k). Consequently, the correlation functions $\rho_k^{\mathrm{U}(n)}$, $1 \leq k \leq n$, are determinants based on the same kernel :

$$\rho_k^{\mathrm{U}(n)}(\theta_1, \dots, \theta_n) = \det_{k \times k} \left(K^{\mathrm{U}(n)}(\theta_i - \theta_j) \right), \; K^{\mathrm{U}(n)}(\theta) = \frac{1}{2\pi} \frac{\sin(n\theta/2)}{\sin(\theta/2)}.$$

In an easier way than for the GUE, the limiting sine kernel again appears

$$\frac{2\pi}{n} K^{\mathrm{U}(n)} \left(\frac{2\pi\theta}{n} \right) \underset{n \to \infty}{\longrightarrow} K(\theta).$$

This microscopic description of the fermionic aspect of the eigenvalues also appears in a number theoretic context, as considered in the next section.

1.4 Repulsion of the zeta zeros

Being simply the Fourier transform of an interval, the sine kernel (12) appears in many different contexts in mathematics and physics. However it was a striking result when Montgomery discovered in the early 70's that it describes the pair correlation of the zeta zeros. During tea time in Princeton he mentioned his result to Dyson, who immediately recognized the limiting pair correlation r_2 for eigenvalues of the GUE, (13). This unexpected result gave new insight into the Hilbert-Pólya suggestion that the zeta zeros might linked to eigenvalues of a self-adjoint operator acting on a Hilbert space.

The Random-Matrix connection was tested numerically by Odlyzko [38] and found to provide a remarkably accurate model of the data. For example, supposing that all orders correlations of the ζ zeros coincide with determinants of the sine kernel, one expects that the histogram of the normalized spacings between zeros converge to the distribution function of the same asymptotics related to the eigenvalues of the GUE or unitary group.

[3] i.e. the unique left (or right) translation invariant measure on the compact group $\mathrm{U}(n)$: if x has distribution $\mu_{\mathrm{U}(n)}$ then for any fixed $a \in \mathrm{U}(n)$

$$ax \overset{\text{law}}{=} x.$$

More precisely, we write as previously $1/2 \pm \mathrm{i}\gamma_n$ for the zeta zeros counted with multiplicity, assume the Riemann hypothesis and the order $\gamma_1 \leq \gamma_2 \leq \dots$ Let $\omega_n = \frac{\gamma_n}{2\pi}\log\frac{\gamma_n}{2\pi}$. From (2) we know that $\delta_n = \omega_{n+1} - \omega_n$ has a mean value 1 as $n \to \infty$, and its repartition function is expected to converge to

$$\frac{1}{n}|\{k \leq n : \delta_k < s\}| \underset{n\to\infty}{\longrightarrow} -\partial_s \det(\mathrm{Id} - K_{(0,s)}),$$

where $K_{(0,s)}$ is the convolution operator acting on $\mathrm{L}^2(0,s)$ with kernel K. This comes from the inclusion-exclusion principle linking free intervals and correlation functions, in which the determinantal structure leads to a Fredholm determinant (see e.g. [1]).

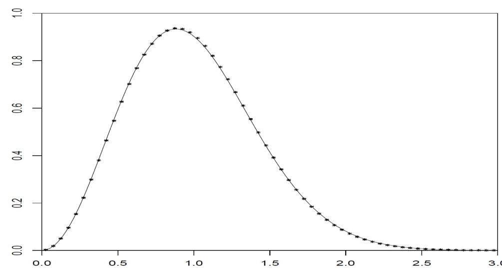


FIG. 3 – The distribution function of asymptotic gaps between eigenvalues ($\partial_s \det(\mathrm{Id} - K_{(0,s)})$) compared with the histogram of gaps between normalized ζ zeros, based on a billion zeros near $\#1.3 \cdot 10^{16}$ (by Odlyzko).

What exactly did Montgomery prove? Rather than mean spacings, a more precise understanding of the zeta zeros interactions relies on the study, as $t \to \infty$, of the spacings distribution function

$$\frac{1}{\mathcal{N}(t)}|\{(n,m) \in [\![1, \mathcal{N}(t)]\!]^2 : \alpha < \omega_n - \omega_m < \beta, n \neq m\}|,$$

where $\mathcal{N}(t)$ is the number of zeros till height t, and more generally the operator

$$r_2(f,t) = \frac{1}{\mathcal{N}(t)} \sum_{1 \leq j,k \leq \mathcal{N}(t), j \neq k} f(\omega_j - \omega_k).$$

If the $\omega_k' s$ were asymptotically independently distributed (up to ordering), $r_2(f,x)$ would converge to $\int_{\mathbb{R}} f(y)\mathrm{d}y$ as $x \to \infty$. That this is not the case follows from an important theorem due to Montgomery [34] :

Theorem 1.8. *Assume the Riemann hypothesis. Suppose f is a test function with the following property : its Fourier transform*[4] *is $\mathscr{C}^\infty$ and supported in $(-1,1)$. Then*

$$r_2(f,t) \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}} f(y) r_2(y) \mathrm{d}y,$$

where $r_2(y) = 1 - \left(\frac{\sin(\pi y)}{\pi y}\right)^2$, as for Wigner or unitary random matrices.

An important conjecture due to Montgomery asserts that the above result holds with no condition on the support of the Fourier transform, but weakening the restriction even to $\operatorname{supp} \hat{f} \subset (-1-\varepsilon, 1+\varepsilon)$ for some $\varepsilon > 0$ seems out of reach with known techniques. The Montgomery conjecture would have important consequences for example in terms of second moments of primes in short intervals [35].

[4] Contrary to the Weil and Selberg formulas (5) and (7), the chosen normalization here is $\hat{f}(x) = \int_{-\infty}^{\infty} f(y) e^{-\mathrm{i}2\pi x y} \mathrm{d}y$

Sketch of proof of Montgomery's Theorem. Consider the function

$$F(\alpha,t)=\frac{1}{\frac{t}{2\pi}\log t}\sum_{0\leq\gamma,\gamma'\leq t}t^{\mathrm{i}\alpha(\gamma-\gamma')}\frac{4}{4+(\gamma-\gamma')^2}.$$

This is the Fourier transform of the normalized spacings, up to the factor $4/(4+(\gamma-\gamma')^2)$. This function naturally appears when counting the second order moments

$$\int_0^t|G(s,t^\alpha)|^2\mathrm{d}s=F(\alpha,t)t\log t+\mathrm{O}(\log^3 t),\ G(s,x)=2\sum_\gamma\frac{x^{\mathrm{i}\gamma}}{1+(s-\gamma)^2}. \tag{14}$$

As G is a linear functional of the zeros, it can be written as a sum over primes by an appropriate explicit formula[5] like (5) :

$$\begin{aligned}G(s,x)=-\sqrt{x}\left(\sum_{n\leq x}\Lambda(n)\left(\frac{x}{n}\right)^{-\frac{1}{2}+\mathrm{i}s}+\sum_{n>x}\Lambda(n)\left(\frac{x}{n}\right)^{\frac{3}{2}+\mathrm{i}s}\right)\\ +x^{-1+\mathrm{i}s}(\log(|s|+2)+\mathrm{O}(1))+\mathrm{O}\left(\frac{\sqrt{x}}{|s|+2}\right),\end{aligned}$$

a fundamental formula due to Montgomery, which requires the Riemann hypothesis to yield the error term quoted. The moment (14) can therefore be expanded as a sum over primes, and the Montgomery-Vaughan inequality (Theorem 1.9) leads to

$$\int_0^t|G(s,t^\alpha)|^2\mathrm{d}s=(t^{-2\alpha}\log t+\alpha+\mathrm{o}(1))t\log t.$$

These asymptotics can be proved by the Montgomery Vaughan inequality, but only in the range $\alpha\in(0,1)$, which explains the support restriction in the hypotheses. Gathering both asymptotic expressions for the second moment of G yields $F(\alpha,t)=t^{-2\alpha}\log t+\alpha+\mathrm{o}(1)$. Finally, by the Fourier inverse formula,

$$\frac{1}{\frac{t}{2\pi}\log t}\sum_{0\leq\gamma,\gamma'\leq t}f\left((\gamma-\gamma')\frac{\log t}{2\pi}\right)\frac{4}{4+(\gamma-\gamma')}=\int_{\mathbb{R}}F(\alpha,t)\hat{f}(\alpha)\mathrm{d}\alpha.$$

If $\operatorname{supp}\hat{f}\subset(-1,1)$, this is approximately

$$\begin{aligned}\int_{\mathbb{R}}\hat{f}(\alpha)(t^{-2|\alpha|}+\alpha+\mathrm{o}(1))\mathrm{d}\alpha&=\int_{\mathbb{R}}e^{-2|\alpha|}\hat{f}(\alpha/\log t)\mathrm{d}\alpha+\int_{\mathbb{R}}\alpha\hat{f}(\alpha)\mathrm{d}\alpha\\ &=\hat{f}(0)+\int_{\mathbb{R}}\alpha\hat{f}(\alpha)\mathrm{d}\alpha+\mathrm{o}(1)=\int_{\mathbb{R}}f(x)\left(1-\left(\frac{\sin\pi x}{\pi x}\right)^2\right)\mathrm{d}x+\mathrm{o}(1)\end{aligned}$$

by the Plancherel formula. □

Theorem 1.9. *Let (a_r) be complex numbers, (λ_r) distinct real numbers and $\delta_r=\min_{s\neq r}|\lambda_r-\lambda_s|$. Then*

$$\frac{1}{t}\int_0^t|\sum_r a_re^{\mathrm{i}\lambda_r s}|^2\mathrm{d}s=\sum_r|a_r|^2\left(1+\frac{3\pi\theta}{t\delta_r}\right)$$

[5]The factor $4/(4+(\gamma-\gamma')^2)$ gives convergence properties necessary to the explicit formula.

for some $|\theta| < 1$. *In particular,*

$$\int_0^t \left| \sum_{n=1}^{\infty} \frac{a_n}{n^{\mathrm{i}s}} \right|^2 \mathrm{d}s = t \sum_{n=1}^{\infty} |a_n|^2 + \mathrm{O}\left(\sum n |a_n|^2 \right).$$

Montgomery's result has been extended in the following directions. Hejhal [27] proved that the triple correlations of the zeta zeros coincide with those of large Haar-distributed unitary matrices, and Rudnick and Sarnak [42] then showed that all correlations agree. These results are all restricted by the condition that the Fourier transform of f is supported on some compact set. To state the Rudnick-Saenak result, we note as in [42] :

- $\mathcal{E}_t = \{\omega_i : i \leq \mathcal{N}(t)\}$;
- f is a translation invariant function from $\mathbb{R}^n$ to $\mathbb{R}$ ($f(x+t(1,\dots,1) = f(x))$), symmetric and rapidly decreasing[6] on $\sum_1^k x_i = 0$;
- $r_n(f,t) = \frac{n!}{\mathcal{N}(t)} \sum_{S \subset \mathcal{E}_t, |S|=n} f(S)$, generalizing the previous definition of $r_2(f,t)$.

Theorem 1.10. *Assume the Riemann hypothesis*[7] *and that the Fourier transform of* f *is supported in* $\sum_1^n |\xi_j| < 2$. *Then*

$$r_n(f,t) \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^n} f(x) \det_{n\times n}\left(\frac{\sin \pi(x_i - x_j)}{\pi(x_i - x_j)} \right) \delta_{x_1+\dots+x_n} \mathrm{d}x_1 \dots \mathrm{d}x_n.$$

Sketch of proof. The method employed by Rudnik and Sarnak makes use of smoothed statistics, namely

$$c_n(f,t,h) = \sum_{j_1,\dots,j_n} h\left(\frac{\gamma_{j_1}}{t}\right) \dots h\left(\frac{\gamma_{j_n}}{t}\right) f\left(\frac{\log t}{2\pi}\gamma_{j_1}, \dots, \frac{\log t}{2\pi}\gamma_{j_n} \right),$$

not assuming here that the indexes are necessarily distinct. This allows the use of two important ingredients :

- a Fourier transform to convert the nth-order statistics to linear ones :

$$c_n(f,t,h) = \int_{\mathbb{R}^n} \prod_{k=1}^{n} \sum_{j_k} h\left(\frac{\gamma_{j_k}}{t}\right) t^{-\mathrm{i}\gamma_{j_k}\xi_k} \mathrm{d}\mu(\xi), \qquad (15)$$

  where $\mathrm{d}\mu(\xi) = \Phi(\xi)\delta_{\xi_1+\dots+\xi_n}\mathrm{d}\xi_1 \dots \mathrm{d}\xi_n$ is the Fourier transform of f ;
- Weil's explicit formula (5), or a variant, to transfer linear statistics over zeros to linear statistics over primes :

$$\sum_\gamma h(\gamma) = h\left(\frac{i}{2}\right) + h\left(-\frac{i}{2}\right) + \frac{1}{2\pi}\int_{\mathbb{R}} h(r)\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + \mathrm{i}r\right) + \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} - \mathrm{i}r\right) \right) \mathrm{d}r$$
$$- \sum_n \frac{\Lambda(n)}{\sqrt{n}} \hat{h}(\log n) + \frac{\Lambda(n)}{\sqrt{n}} \hat{h}(-\log n). \qquad (16)$$

---

[6] i.e. faster than any $|x|^{-\lambda}$ for any $\lambda > 0$

[7] An unconditional result holds with smoothed test functions.

Substituting (16) into (15) and expanding the product, we end up with a sum of terms like

$$c_{r,s}(t) = \sum_{\mathbf{n}} \frac{\Lambda(n_1)\dots\Lambda(n_{r+s})}{\sqrt{n_1\dots n_{r+s}}} t^n$$

$$\int_{\mathbb{R}^n} \prod_{j=1}^{r} \hat{h}(t((\log t)\xi_j + \log n_j)) \prod_{j=r+1}^{r+s} \hat{h}(t((\log t)\xi_j - \log n_j)) \prod_{j>r+s} \hat{h}(t(\log t)\xi_j)\mathrm{d}\mu(\xi).$$

As $\sum|\xi_j| < 2$, one can use the Montgomery Vaughan inequality Theorem 1.9 to get the correct asymptotics : in the above sum the main contribution comes from choices of $\mathbf{n}$ such that

$$n_1 \dots n_r = n_{r+1} \dots n_{r+s}, \tag{17}$$

i.e. the diagonal elements. The Van Mongoldt function being supported on prime powers, the main contribution comes from the choice of prime n_j's, which implies $r = s$ by (17). We are therefore led to the asymptotics of

$$c_{r,r}(t) = \frac{t}{2\pi \log^{2r-1} t} \int_{\mathbb{R}} h(r)^n \mathrm{d}r \sum_{p_1,\dots,p_r \ll t} \sum_{\sigma \in \mathscr{S}_r} \frac{\log^2 p_1 \dots \log^2 p_r}{p_1 \dots p_r}$$
$$\Phi\left(-\frac{\log p_1}{\log t}, \dots, \frac{\log p_r}{\log t}, \frac{\log p_{\sigma(1)}}{\log t}, \dots, \frac{\log p_{\sigma(r)}}{\log t}, 0, \dots, 0\right),$$

where $\mathscr{S}_r$ is the symmetric group with r elements. The equivalent of the above sum can be calculated thanks to the prime number theorem and integration by parts, leading to the estimate

$$c_n(f,t,h) \underset{t\to\infty}{\sim} \frac{t\log t}{2\pi} \int_{\mathbb{R}} h(r)^n \mathrm{d}r$$
$$\left(\Phi(0) + \sum_{r=1}^{\lfloor n/2 \rfloor} \sum \int |v_1| \dots |v_r| \Phi(v_1 e_{i_1,j_1}, \dots, v_r e_{i_r,j_r}, 0, \dots, 0) \mathrm{d}v_1 \dots \mathrm{d}v_r \right), \tag{18}$$

where the sum is over all choices of pairs of disjoint indices in $[\![1, n]\!]$ and $e_{i,j} = e_i - e_j$, (e_i) being an orthonormal basis of $\mathbb{C}^n$.

At this point, it is not clear how this is related to determinants of the sine kernel. This is a purely combinatorial problem : by inclusion-exclusion the asymptotics of $r_n(f,t,h)$ can be deduced from those of $c_m(f,t,h)$, for all m. Then it turns out that when writing

$$r_n(f,t,h) = \frac{t\log t}{2\pi} \int_{\mathbb{R}} h(r)^n \int_{\mathbb{R}^n} r(v)\Phi(v)\mathrm{d}v(1 + \mathrm{o}(1)),$$

the function r is exactly the Fourier transform of the determinant of the sine kernel, $\det_{n\times n} K(x_i - x_j)$.

For this last step, another way to proceed consists in making the same reasoning by replacing the zeta zeros by eigenvalues of a unitary matrix u, and computing expectations with respect to the Haar measure. The Fourier transform and explicit formula (rewriting linear statistics of eigenvalues as linear sums of $(\mathrm{Tr}(u^k))_k$) still hold. Diaconis and Shahshahani [20] proved that these traces converge in law to independent normal complex gaussians as $n \to \infty$. This independence is equivalent to performing the above diagonal approximation (17) and allows one to get formula (18) in the context of random matrices. We independently know that the eigenvalues correlations are described by the sine kernel, which completes the proof. □

The scope of this analogy needs to be moderated : following [4, 8, 5], we will see in the next section that beyond leading order the two-point correlation function depends on the positions of the low ζ zeros, something that clearly contrasts with random matrix theory.

Moreover, Rudnick and Sarnak proved that the same fermionic asymptotics hold for any primitive L-function. However, we cannot expect that it holds for any L-function, because for example, for distinct primitive characters, the zeros of L_χ and $\mathrm{L}_{\chi'}$ have no known link, so for the product of these L-functions, the zeros look like the superposition of two independent determinantal point processes. Systems with independent versus repelling eigenvalues are discussed in the next section.

2 Quantum chaology

Quantum chaos is concerned with the study the quantum mechanics of classically chaotic systems. In reality, a quantum system is much less dependent on the initial conditions than a classical chaotic one, where orbits are generally divergent. This is the reason why M. Berry proposed the name *quantum chaology* instead of *quantum chaos*.

The statistics found for the ζ zeros can, in this context, be seen in a more general framework. Indeed, eigenvalue repulsion is conjectured to appear in the statistics of generic chaotic systems. In the same way as the appearance of the sine kernel in the description of the statistics of the ζ zeros is proved using the explicit formula, the Bohigas-Giannoni-Schmidt conjecture is intimately linked to a semiclassical asymptotic generalization of Selberg's trace formula due to Gutzwiller. When going from a trace formula to correlations, one deals with diagonal and non-diagonal terms (i.e. repeated or distinct orbits), and their relative magnitude is crucial. We will discuss below to which extent the diagonal terms dominate, and how to estimate the contribution of non-diagonal ones.

2.1 The Berry-Tabor and Bohigas-Giannoni-Schmit conjectures

One of the goals of quantum chaology is to exhibit characteristic properties of quantum systems which, in the semiclassical limit[8], reflect the regular or chaotic aspects of the underlying classical dynamics. For example, how does classical mechanics contribute to the distribution of the eigenvalues and the amplitudes of the eigenfunctions when the de Broglie wavelength tends to 0 ?

The examples we consider are two-dimensional quantum billiards[9]. For some billiards, the classical trajectories are integrable (regular) and for others they are

[8] The semiclassical limit corresponds to $\hbar \to 0$ in the Schrödinger equation ; in many cases, including the examples considered here, this corresponds to the high-energy limit.

[9] A billiard is a compact connected set with nonempty interior, with a generally piecewise regular boundary, so that the classical trajectories are straight lines reflecting with equal angles of incidence and reflection

chaotic. On the quantum side, the standing waves are described by the Helmholtz equation

$$-\frac{\hbar^2}{2m}\Delta\psi_n = \lambda_n\psi_n,$$

where the spectrum is discrete as the domain is compact, with ordered eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \dots$, and appropriate Dirichlet or Neumann boundary conditions. The questions about quantum billiards one is interested in include : how does $|\psi_n|^2$ get distributed in the domain and what is the asymptotic distribution of the λ_n's as $n \to \infty$?

One fundamental result due to Schnirelman [43] states that the quantum eigenfunctions become equidistributed with respect to the Liouville measure[10] ν, as $n \to \infty$, along a subsequence $(n_k)_{k\geq 0}$ of density one : for any measurable set I in the domain $\mathcal{D}$

$$\lim_{k\to\infty} \frac{\int_I |\psi_{n_k}|^2 \mathrm{d}x\mathrm{d}y}{\int_{\mathcal{D}} |\psi_{n_k}|^2 \mathrm{d}x\mathrm{d}y} = \frac{\nu(I)}{\nu(\mathcal{D})}.$$

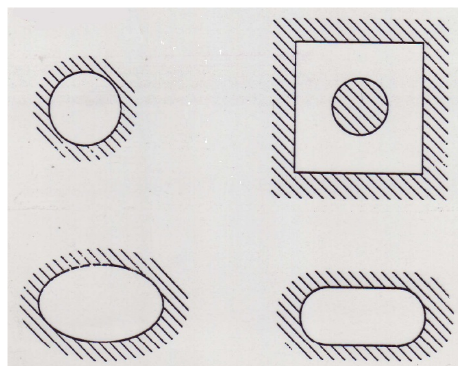


FIG. 4 – Regular (left) and chaotic (right) billiards. Upper right : Sinai's billiard

This is referred to as *quantum ergodicity*. A stronger equipartition notion, *quantum unique ergodicity* [42], states that the above limit holds over $\mathbb{N}$, with no exceptional eigenfunctions. This is proved in very few cases. The systems where it has been proved include holomorphic cusp forms (related to billiards on $\mathbb{H}$), thanks to the work of Holowinsky and Soundararajan [28]. To satisfy quantum unique ergodicity, a system needs to avoid the problem of *scars* : for some chaotic systems, some eigenfuntions (a negligible fraction of them) present an enhanced modulus near the short classical periodic orbits.

In great generality, according to the *semiclassical eigenfunction hypothesis*, the eigenstates should concentrate on those regions explored by a generic orbit as $t \to \infty$: for integrable systems the motion concentrates onto invariant tori while for the ergodic ones the whole energy surface is filled in a uniform way.

Concerning eigenvalue statistics, the situation is still complicated and somehow mysterious : there is a conjectural dichotomy between the chaotic and integrable cases.

First, in 1977, Berry and Tabor [3] put forward the conjecture that for a generic integrable system[11] the eigenvalues have the statistics of a Poisson point process, in the semiclassical limit. More precisely, by Weyl's law, we know that the number of such eigenvalues up to λ is

$$|\{i : \lambda_i \leq \lambda\}| \underset{\lambda\to\infty}{\sim} \frac{\mathrm{area}(\mathcal{D})}{4\pi}\,\lambda. \tag{19}$$

[10] i.e. the Lebesgue mesure in our Euclidean case

[11] There are exceptions, obvious or less obvious, many of them already known by Berry and Tabor[3], which is the reason why one expects the Poissonian behavior for *generic* systems.

To analyze the correlations between eigenvalues, consider the point process

$$\chi^{(n)} = \frac{1}{n}\sum_{i\leq n}\delta_{\frac{4\pi}{\mathrm{area}(\mathcal{D})}(\lambda_{i+1}-\lambda_i)},$$

which has an expectation equal to 1 from (19). By the expected limiting Poissonian behavior, the spacing distribution converges to an exponential law : for any $I\subset\mathbb{R}^+$

$$\chi^{(n)}(I) \underset{n\to\infty}{\longrightarrow} \int_I e^{-x}\mathrm{d}x. \tag{20}$$

The limiting independence of the λ_j's also implies a variance of order n, like for any central limit theorem, in the above convergence. Note that the Berry-Tabor conjecture was rigorously proved for many integrable systems in the sense of *almost all* systems in certain families. One unconditional result concerns some fixed shifts on the torus : Marklof [32] proved that for a free particle on $\mathbb{T}^k$ with flux lines of strength $\alpha=(\alpha_1,\dots,\alpha_k)$, if α is diophantine of type $\kappa<(k-1)/(k-2)$ and the components of $(\alpha,1)$ are linearly independent over $\mathbb{Q}$, then the pair correlation of eigenvalues is asymptotically Poissonian.

In the chaotic case, the situation is radically different, the variance when counting the energy levels is believed to be of order $\log n$, so much less than in (20) : the eigenvalues are supposed to repel each other and their statistics are conjectured to be similar to those of a random matrix, from an ensemble depending on the symmetries properties of the system (e.g. time-reversibility). This is known as the Bohigas-Giannoni-Schmidt Conjecture [9] (but see also [3]).

Numerical experiments were performed in [9] giving a correspondence between the eigenvalue spacings statistics for Sinai's billiard and those of the Gaussian Orthogonal Ensemble[12]. Dyson's reaction to these conjecture and experiments was the following[13].

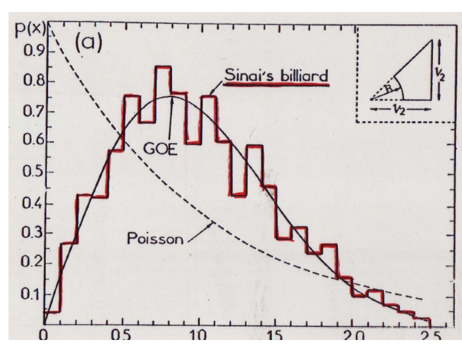


FIG. 5 – Energy levels for Sinai's billiard compared to those of the Gaussian Orthogonal Ensemble and Poissonian statistics.

This is a beautiful piece of work. It is extraordinary that such a simple model shows the GOE behavior so perfectly. I agree completely with your conclusions. I would say that the result is not quite surprising but certainly unexpected...I once suggested to a student at Haverford that he build a microwave cavity and observe the resonances to see whether they follow the GOE distribution. So far as I know, the experiment was never done...I always thought the cavity would have to be a complicated shape with many angles. I did not imagine that something as simple as the Sinai region would work.

A theoretical understanding of this conjecture proposed in [2] is related to correlations between classical periodic orbits, via the Gutzwiller trace formula explained

[12] i.e. the same eigenvalues spacings as for a random symmetric matrix with gaussian entries.

[13] In a letter to O. Bohigas, 1983.

in the next section. Our purpose consists in understanding the role of orbits of the classical motion to give insight into the derivation of the correlations of the ζ zeros [6, 7, 7].

2.2 Periodic Orbit Theory

Consider a set of positive eigenvalues (λ_n) and the counting function

$$\mathrm{N}(\lambda) = \sum_{n=1}^{\infty} \mathbb{1}_{\lambda_n<\lambda}.$$

In typical situations, this can be decomposed into a mean term and fluctuations,

$$\mathrm{N}(\lambda) = \langle \mathrm{N}(\lambda)\rangle + \mathrm{N}^{\mathrm{fl}}(\lambda).$$

For example, in the case of a quantum billiard on a domain $\mathcal{D}$, as previously discussed, the mean term is independent of whether the classical dynamics is regular or chaotic and is given by

$$\langle \mathrm{N}(\lambda)\rangle = \frac{\mathrm{area}(\mathcal{D})}{4\pi}\lambda,$$

as shown by (19) and the fluctuating part, with mean zero, encodes independence (integrable) or repulsion (chaotic) for the energy levels. In another context, when counting the imaginary parts of the non-trivial ζ zeros, formula (2) implies

$$\begin{aligned}\langle \mathrm{N}(t)\rangle &= \frac{t}{2\pi}\log\frac{t}{2\pi e} + \frac{7}{8} + \mathrm{O}\left(\frac{1}{t}\right),\\ \mathrm{N}^{\mathrm{fl}}(t) &= \frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+\mathrm{i}t\right) = \frac{1}{\pi}\Im\left(\log\zeta\left(\frac{1}{2}+\mathrm{i}t\right)\right).\end{aligned}$$

The fact that this fluctuating part has mean zero can be seen as a byproduct of the central limit theorem (38).

The Euler product expression for ζ is not known to hold for $\sigma \in (1/2, 1)$ (this is related to the Riemann hypothesis), but we write formally

$$\begin{aligned}\mathrm{N}^{\mathrm{fl}}(t) &= -\frac{1}{\pi}\sum_{p}\Im\left(\log\left(1-\frac{e^{-\mathrm{i}t\log p}}{\sqrt{p}}\right)\right)\\ &= -\frac{1}{\pi}\sum_{\mathcal{P},\mathbb{N}^*}\frac{e^{-\frac{1}{2}m\log p}}{m}\sin(tm\log p).\end{aligned} \tag{21}$$

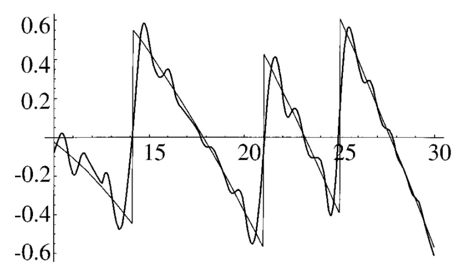


FIG. 6 – $\mathrm{N}^{\mathrm{fl}}(t)$ (thin line) compared with the truncated expansion (thick line) from (21) with the first 50 primes and all m.

As shown in Figure 6, truncating this expansion provides meaningful results.

We want to place the above fluctuation formulae in a more general context. Consider a dynamical system with coordinates $\mathbf{q} = (q_1, \dots, q_d)$ and momenta $\mathbf{p} = (p_1, \dots, p_d)$. The trajectories are generated by a Hamiltonian $\mathrm{H}(\mathbf{q}, \mathbf{p})$. On the quantum side, $\mathbf{q}$ and $\mathbf{p}$ are operators with commutator $[\mathbf{q}, \mathbf{p}] = \mathrm{i}\hbar$, so H is an operator whose eigenvalues are the quantum energy levels.

For quantum billiards, H is independent of $\mathbf{q}$ in $\mathcal{D}$. We are interested in the situation where the energy is the only conserved quantity and neighboring trajectories diverge exponentially : the system is chaotic.

As seen in Section 1, the explicit formula (5) states that the ζ zeros have a distribution formally similar to the the eigenvalues of the hyperbolic Laplacian, through the Selberg trace formula (7). This admits a semiclassical (i.e. asymptotic) generalization, originally derived in [36]. For a periodic orbit p, we denote the action by $\mathrm{S}_p(\lambda) = \oint \mathbf{p} \cdot \mathrm{d}\,\mathbf{q}$ and the period by $\mathrm{T}_p = \partial_\lambda \mathrm{S}_p$. The monodromy matrix M_p describes the exponential divergence of deviations from p of nearby geodesics. The Maslov index μ_p is related to the winding number of the invariant Lagrangian (stable and unstable) manifolds around the orbit : it describes the topological stability. The Maslov index of the m-repetition of the orbit p is equal to $m\mu_p$.

Gutwiler's trace formula. *With the preceding notation,*

$$\mathrm{N}^{\mathrm{fl}}(\lambda) \underset{\lambda\to\infty}{\sim} \frac{1}{\pi} \sum_{\mathcal{P},\mathbb{N}^*} \frac{\sin\left(m\,\mathrm{S}_p(\lambda) - \frac{m\pi\mu_p}{2}\right)}{m\sqrt{|\det(\mathrm{M}_p^m - \mathrm{Id})|}}, \tag{22}$$

where $\mathcal{P}$ is the set of primitive orbits and m is the index of their repetitions.

To some extent, this formula should be considered as natural.

- The energy levels counted by N are associated with stationary states, i.e. time-independent objects. Their asymptotics correspond to the phase space structure invariant under translations along geodesics, by the correspondence principle. In our chaotic situation, there are two types of invariant manifolds, the whole surface (by ergodicity), leading to the term $\langle N(\lambda)\rangle$ and the periodic orbits which correspond to the fluctuations $\mathrm{N}^{\mathrm{fl}}(\lambda)$.
- The exactness of the trace formula for manifolds of constant negative curvature (Selberg's trace formula) is analogous to the exact formula for the heat kernel in the Euclidean space. In the more general context of Riemannian manifolds, the heat kernel estimates are known only for short times and in terms of the geodesic distance : $p(x,y,t) \underset{t\to 0}{\sim} c\exp(-\ell(x,y)^2/2t)/t^{d/2}$, where the constant c involves the deviations from the geodesic via the Van Vleck-Morette determinant, analogously to $\det(\mathrm{M}_p^m - \mathrm{Id})$ in the Gutzwiller trace formula.

Sketch of proof. Writing $d(\lambda) = \mathrm{d}\,\mathrm{N}(\lambda)/\mathrm{d}\lambda$, we begin in the same way as for the Selberg trace formula, writing

$$d(\lambda) = -\frac{1}{\pi} \int \Im\left(G^{(\lambda)(\mathbf{x},\mathbf{x})}\right) \mathrm{d}\,\mathbf{x},$$

where $G^{(\lambda)}$ is the Green function associated with the energy λ. The mean eigenvalue density $\langle d(\lambda)\rangle$ corresponds to the small (minimal distance) trajectories between $\mathbf{x}$ and $\mathbf{y}$ as $\mathbf{y} \to \mathbf{x}$, and the fluctuating part $d^{\mathrm{fl}}(\lambda)$ is related to all other geodesics between $\mathbf{x}$ and itself, for example all repeated maximal circles in the spherical situation. A key assumption about the Green function is that it admits the expansion

$$G^{(\lambda)}(\mathbf{x},\mathbf{y}) = \sum_{\text{geodesics}} A(\mathbf{x},\mathbf{y}) e^{\mathrm{i}\,\mathrm{S}(\mathbf{x},\mathbf{y})/\hbar},$$

where A can be developed as a series in $\hbar$, the sum is over all geodesics from $\mathbf{x}$ to $\mathbf{y}$, and $\mathrm{S}(\mathbf{x},\mathbf{y}) = \int \mathbf{p} \cdot \mathrm{d}\,\mathbf{q}$ depends on the trajectory and λ. This formula is justified by inserting $A(\mathbf{x},\mathbf{y})e^{\mathrm{i}\,\mathrm{S}(\mathbf{x},\mathbf{y})/\hbar}$ into the Schrödinger equation. Consequently,

$$d^{\mathrm{fl}}(\lambda) = \frac{1}{\pi} \int \Im \left(\sum_{\text{non-trivial geodesics}} A(\mathbf{x},\mathbf{x}) e^{\mathrm{i}\,\mathrm{S}(\mathbf{x},\mathbf{x})/\hbar} \right) \mathrm{d}\,\mathbf{x} \,.$$

A saddle point approximation can be performed as $\hbar \to 0$. On any critical point, $(\partial_{\mathbf{x}} \mathrm{S} + \partial_{\mathbf{y}} \mathrm{S})_{\mathbf{x}=\mathbf{y}} = \mathbf{0}$, but $\partial_{\mathbf{x}} \mathrm{S} = \mathbf{p}_f$ and $\partial_{\mathbf{y}} \mathrm{S} = \mathbf{p}_i$, the momenta at the final and initial points respectively. Consequently, on the saddle, the momenta must be identical at the beginning and the end of the geodesic : the trajectory is periodic. The second derivatives, leading to the constant coefficients in the saddle point approximation, are related to the monodromy matrix, corresponding to the linear approximation between initial and final perturbations along the periodic orbit p :

$$\mathrm{d} \begin{pmatrix} \mathbf{q}_f \\ \mathbf{p}_f \end{pmatrix} = \mathrm{M}_p \, \mathrm{d} \begin{pmatrix} \mathbf{q}_i \\ \mathbf{p}_i \end{pmatrix} .$$

Moreover, when performing the saddle-point method, the Maslov index appears because it counts, roughly speaking, the number of caustics along the trajectory. All results together, with periodic orbits seen as repetitions of primitive periodic orbits, explain the origin of the main terms in (22). □

An approximation of the determinant can be performed for long orbits, in terms of the Liapunov (instability) exponent of the orbit, noted λ_p, and the (large) period $\mathrm{T}_p = \partial_\lambda \mathrm{S}_p : \det(\mathrm{M}_p^m - \mathrm{Id}) \approx e^{m\lambda_p \mathrm{T}_p}$, so the Gutzwiller trace formula takes the form

$$\mathrm{N}^{\mathrm{fl}}(\lambda) \underset{\lambda\to\infty}{\sim} \frac{1}{\pi} \sum_{\mathcal{P},\mathbb{N}^*} \frac{e^{-\frac{1}{2} m\lambda_p \mathrm{T}_p}}{m} \sin\left(m\, \mathrm{S}_p(\lambda) - \frac{m\pi\mu_p}{2} \right) . \tag{23}$$

A comparison between formulas (21) and (23) yields the following formal definition of action, period and stability in the prime number context [5].

Eigenvalues	Quantum energy levels	Zeta zeros
Asymptotics	$\hbar \to 0$	$t \to \infty$
Actions	$\frac{m\,\mathrm{S}_p}{\hbar}$	$mt \log p$
Periods	$m\,\mathrm{T}_p$	$m \log p$
Stabilities	λ_p	1

2.3 Diagonal approximation

The link between the eigenvalue counting functions discussed above and the correlation functions is formally given by

$$\begin{aligned} r_n^{(\lambda)}(x_1, \dots, x_n) &= \langle d(\cdot + x_1) \dots d(\cdot + x_n) \rangle \\ &= \langle d \rangle^n + r_n^{(\lambda,\mathrm{diag})}(x_1, \dots, x_n) + r_n^{(\lambda,\mathrm{off})}(x_1, \dots, x_n), \end{aligned} \tag{24}$$

where $d(\lambda) = \frac{\partial\, \mathrm{N}(\lambda)}{\partial \lambda}$ is the eigenvalues density, $r_n^{(\lambda)}$ is the correlation function of order n when considering eigenvalues up to height λ, and the terms $r_n^{(\lambda,\mathrm{diag})}, r_n^{(\lambda,\mathrm{off})}$ will be made explicit in the next few lines. The above formula makes sense once integrated with respect to a smooth enough test function, where for convenience no repetition between distinct eigenvalues is performed :

$$r_n^{(\lambda)}(f) := \frac{n!}{\mathrm{N}(\lambda)} \sum_{S\subset\mathcal{E}_\lambda, |S|=n} f(S) = \int r_n^{(\lambda)}(x) f(x)\mathrm{d}x,$$

where $\mathcal{E}_\lambda$ is the set of eigenvalues up to height λ. (24) together with Gutzwiller's trace formula (22) allows one to calculate the correlation functions from the density, including for the ζ zeros, as in [6, 7]. We describe this approach, and show how it provides a heuristic justification for Montgomery's Conjecture, and also how it yields lower order corrections to the random matrix limit for all orders of correlation functions. In order to be explicit, we focus on the two-point correlation function, $n = 2$.

The results from the previous paragraph can be written, with suitable coefficients $A_{p,m}$'s,

$$d(\lambda) = \langle d(\cdot)\rangle + \sum_{p,m} A_{p,m} e^{\mathrm{i}m\,\mathrm{S}_p(\lambda)/\hbar},$$

which yields, once inserted in (24),

$$r_2^{(\lambda)}(x_1, x_2) \approx \langle d(\cdot)\rangle^2 + \sum_{p_i,m_i} A_{p_1,m_1} \overline{A_{p_2,m_2}} \langle e^{\frac{\mathrm{i}}{\hbar}(m_1\,\mathrm{S}_{p_1}(\cdot+x_1) - m_2\,\mathrm{S}_{p_2}(\cdot+x_2))}\rangle.$$

The terms S_p can be expanded in terms of λ, with first derivative $\partial_\lambda \mathrm{S}_p(\lambda) = \mathrm{T}_p$, so the correlation function takes the form

$$r_2^{(\lambda)}(x_1, x_2) \approx \langle d(\cdot)\rangle^2 + \sum_{p_i,m_i} A_{p_1,m_1} \overline{A_{p_2,m_2}} \langle e^{\frac{\mathrm{i}}{\hbar}(m_1\,\mathrm{S}_{p_1}(\cdot) - m_2\,\mathrm{S}_{p_2}(\cdot))}\rangle . e^{\frac{\mathrm{i}}{\hbar}(m_1\,\mathrm{T}_{p_1}x_1 - m_2\,\mathrm{T}_{p_2}x_2)}. \tag{25}$$

The main difficulty to evaluate this sum consists in an appropriate expectation for $\langle e^{\frac{\mathrm{i}}{\hbar}(m_1\,\mathrm{S}_{p_1}(\cdot) - m_2\,\mathrm{S}_{p_2}(\cdot))}\rangle$. A first approximation consists in keeping only diagonal elements, i.e. orbits with exactly the same action : for distinct trajectories, averaging gives a 0 contribution in the large energy limit.

Let us consider this diagonal approximation in the ζ context (21). The height t dependence disappears when averaging and a direct calculation gives

$$r_2^{(\mathrm{diag})}(x_1, x_2) \approx \frac{\Re}{2\pi^2} \sum_{\mathcal{P},\mathbb{N}^*} \frac{\log^2 p}{p^m} e^{\mathrm{i}(x_1-x_2)m\log p}.$$

The prime number theorem and a series expansion yield

$$r_2^{(\mathrm{diag})}(x_1, x_2) \underset{x_2\to x_1}{\sim} -\frac{1}{2\pi^2(x_1-x_2)^2}.$$

This corresponds to the $r_2^{(\mathrm{diag})}$ part in the following decomposition of the two-point limiting correlation function associated to random matrices :

$$r_2(x) = 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2 = \langle d\rangle^2 + r_2^{(\mathrm{diag})}(x) + r_2^{(\mathrm{off})}(x), \tag{26}$$

$$\langle d\rangle = 1,\ r_2^{(\mathrm{diag})}(x) = -\frac{1}{2(\pi x)^2},\ r_2^{(\mathrm{off})}(x) = \frac{e^{\mathrm{i}2\pi x} + e^{-\mathrm{i}2\pi x}}{4(\pi x)^2}.$$

Alternatively, the right hand side of the sum over primes can be written exactly in terms of the second derivative of $\log\zeta(1+\mathrm{i}(x_1-x_2))$. The pole of the zeta function gives the contribution calculated above from the prime number theorem when $x_1-x_2\to 0$. The structure around the pole, coming from the first few zeros, then gives corrections to the random matrix expression. Hence the statistical properties of the high-lying zeros show universal random-matrix behaviour, with non-universal corrections, which vanish in the appropriate limit, related to the low-lying zeros. This important *resurgence* was discovered in [8] (see also [5] for illustrations and an extensive discussion).

It follows from the fact that the diagonal terms do not give the full expression for the two-point correlation function that the non-diagonal terms are important. More precisely, such terms make no contribution if

$$\frac{1}{\hbar}(m_1\,\mathrm{S}_{p_1}-m_2\,\mathrm{S}_{p_2})\gg 1 \tag{27}$$

in the quantum context. The average gap between lengths of periodic orbits around ℓ is about $\ell e^{-c\ell}$ (the number of orbits grows exponentially with the length). Hence, one expects that the diagonal approximation at the energy level λ can be performed, in the semiclassical limit, for orbits up to height $\ell\ll(\log\lambda)/c$. In particular, one sees with disappointment that this number grows only logarithmically with the energy. In the number theory context, (27) becomes

$$(m_1\log p_1-m_2\log p_2)t\gg 1,$$

and the above discussion shows that we have to tackle the problem of close prime powers.

2.4 Beyond diagonal approximation

Keeping all off-diagonal contributions in (25) for the 2-point correlation of ζ yields

$$r_2^{(t,\mathrm{off})}(x_1,x_2)\approx\sum_{n_1\neq n_2}\frac{\Lambda(n_1)\Lambda(n_2)}{4\pi^2\sqrt{n_1n_2}}\langle e^{\mathrm{i}t\log(n_1/n_2)+\mathrm{i}(x_1\log n_1-x_2\log n_2)}\rangle,$$

where Λ is Van Mangoldt's function. Setting $x=x_1-x_2$ and $n_1=n_2+r$, then expanding all functions in r yields

$$r_2^{(t,\mathrm{off})}(x)\approx\frac{1}{4\pi^2}\sum_{n,r}\frac{\Lambda(n)\Lambda(n+r)}{n}\langle e^{\mathrm{i}t\frac{r}{n}+\mathrm{i}x\log n}\rangle, \tag{28}$$

and the main difficulty therefore becomes the evaluation of the correlations between values of the Van Mangoldt function. No unconditional results are known about it. We make use of the following conjecture, from [25].

The Hardy-Littlewood conjecture. *For any odd r, $\frac{1}{n}\sum_{k\leq n}\Lambda(k)\Lambda(k+r)$ has a limit as $n\to\infty$, equal to*

$$\alpha(r)=c\prod_{p\mid r}\frac{p-1}{p-2},\ \textit{with}\ c=2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right).$$

If r is even, this limit is 0.

Note that the prime number theorem can be stated as

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k\leq n} \Lambda(n) = 1.$$

Therefore, if the functions $\Lambda(k)$ and $\Lambda(k+r)$ were independent in the large k limit, $\alpha(r)$ would be 1. The Hardy-Littlewood conjecture states that this asymptotic independence holds up to arithmetical constraints. Assuming the accuracy of this conjecture yields, from (28),

$$r_2^{(t,\mathrm{off})}(x) \approx \frac{1}{4\pi^2} \sum_{n,r} \alpha(r) e^{\mathrm{i}tr/n + \mathrm{i}x\log n}.$$

This expression can be simplified to get

$$r_2^{(t,\mathrm{off})}(x) \approx \frac{1}{4\pi^2} |\zeta(1+\mathrm{i}x)|^2 \left(\frac{t}{2\pi}\right)^{\mathrm{i}x} \prod_{\mathcal{P}} \left(1 - \frac{1-p^{\mathrm{i}x}}{(p-1)^2}\right).$$

In the limit of small x, $x = \frac{2\pi u}{\log(t/2\pi)}$ (consistent with the required normalization in Montgomery's Theorem 1.8), this expression can be shown to become

$$\frac{e^{\mathrm{i}2\pi u} + e^{-\mathrm{i}2\pi u}}{4(\pi u)^2},$$

the part that was missing in the decomposition (26). This provides strong heuristic support for Montgomery's conjecture. For more details see [6, 7, 8, 5]. Note that once again the approach to the random matrix limit is controlled by $\zeta(1+\mathrm{i}x)$.

Note that the above method, inspired by the periodic orbit theory in quantum chaos, allows one to obtain error terms for the asymptotic pair correlation of the ζ zeros [10]. Taking into account the second order correction to the sine kernel, one gets after scaling

$$r_2^{(t)}(x) = 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2 - \frac{\beta}{\pi^2 \langle d\rangle^2} \sin^2(\pi x) - \frac{\delta}{2\pi^2 \langle d\rangle^3} \sin(2\pi x) + \mathrm{O}(\langle d\rangle^{-4}), \quad (29)$$

where

$$\langle d\rangle = \frac{1}{2\pi} \log\left(\frac{t}{2\pi}\right),$$

and β, δ are numerical constants given by

$$\beta = \gamma_0 + 2\gamma_1 + \sum_p \frac{\log^2 p}{(p-1)^2} \approx 1.573,$$

$$\delta = \sum_p \frac{\log^3 p}{(p-1)^2} \approx 2.315,$$

the γ_k's being the Stieljes constants $\gamma_k = \lim_{m\to\infty}\left(\sum_{j=1}^m \frac{\log^k j}{j} - \frac{\log^{k+1} m}{m+1}\right)$. This second order formula gives remarkably accurate results, as shown in this joint graph.

Finally, the above discussion can be applied to many other ζ statistics. For example, the variance saturation of the counting function of the eigenvalues from the GUE admits a ζ counterpart, observed in Berry's original work [4]. This variance for $\mathcal{N}(t+\delta)-\mathcal{N}(t-\delta)$ has a universal behavior when δ is small enough and an arithmetic influence otherwise.

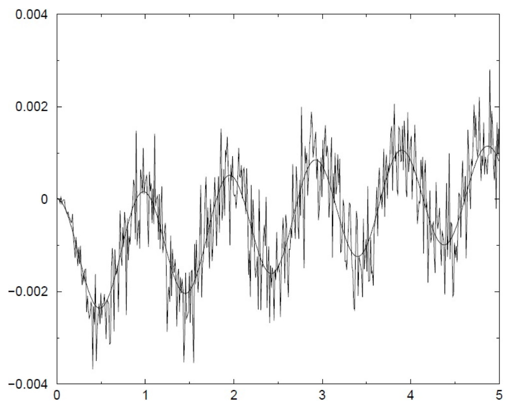


FIG. 7 – Difference $r_2^{(t)}(x) - r_2(x)$, x in $(0,5)$, for $2\cdot 10^8$ Riemann zeros near the 10^23-th zero, graph from [10]. Smooth line from formula (29), oscillating one from Olyzko's numerical data.

3 Macroscopic statistics

In the previous sections, the local fluctuations of zeros of L-functions were shown to be intimately linked to eigenvalues of quantum chaotic systems via random matrix theory. Another type of statistic was considered around 2000, providing even more striking evidence of a connection : at a macroscopic scale, i.e. statistics over all zeros, one also observes a close relationship with Random Matrix Theory. The main example is concerns the moments of ζ.

3.1 Motivations for moments

As already mentioned, amongst the many consequences of the Riemann hypothesis, the Lindelöf conjecture asserts that ζ has a subpolynomial growth along the critical axis : $|\zeta(1/2+\mathrm{i}t)| = \mathrm{O}(t^{\varepsilon})$ for any $\varepsilon > 0$. One of the number theoretic consequences of these bounds would be $p_{n+1} - p_n = \mathrm{O}(p_n^{1/2+\varepsilon})$ for any $\varepsilon > 0$, where p_n is the n th prime number. An apparently weaker conjecture (because it deals with mean values) concerns the moments of ζ : for any $k \in \mathbb{N}$ and $\varepsilon > 0$,

$$I_k(t) = \frac{1}{t}\int_0^t \left|\zeta\left(\frac{1}{2}+\mathrm{i}s\right)\right|^{2k} \mathrm{d}s \ll t^{\varepsilon}.$$

This is actually equivalent to the Lindelöf hypothesis thanks to a good unconditional upper bound on the derivative : $\zeta'(1/2+\mathrm{i}s) = \mathrm{O}(s)$. More precise estimates were proved by Heath-Brown [26] for the following lower bound, and by Soundararajan [47], conditionally on the Riemann hypothesis, for the upper bound :

$$(\log t)^{k^2} \ll I_k(t) \ll (\log t)^{k^2+\varepsilon}$$

for any $\varepsilon > 0$. Unconditional equivalents are known only for $k = 1$ and $k = 2$: Hardy and Littlewood [24] obtained in 1918 that

$$I_1(t) \underset{t\to\infty}{\sim} \sum_{n\leq t} \frac{1}{n} \underset{t\to\infty}{\sim} \log t,$$

and Ingham [29] proved the $k=2$ case

$$I_2(t) \underset{t\to\infty}{\sim} 2\sum_{n\leq t}\frac{d_2(n)^2}{n} \underset{t\to\infty}{\sim} \frac{1}{2\pi^2}(\log t)^4,$$

where the coefficients $d_k(n)$ are defined by $\zeta(s)^k=\sum_n d_k(n)/n^s$, $\Re(s)>1$. Then, a precise analysis led Conrey and Ghosh [16] to conjecture

$$I_3(t) \underset{t\to\infty}{\sim} 43\sum_{n\leq t}\frac{d_3(n)^2}{n}$$

and Conrey and Gonek [18] to

$$I_4(t) \underset{t\to\infty}{\sim} 24024\sum_{n\leq t}\frac{d_4(n)^2}{n}.$$

Is it true that

$$I_k(t) \underset{t\to\infty}{\sim} c_k\sum_{n\leq t}\frac{d_k(n)^2}{n}$$

for some integer c_k, and what should c_k be ? It is known thanks to the behavior of ζ at 1 and Tauberian theorems that

$$\sum_{n\leq t}\frac{d_k(n)^2}{n} \underset{t\to\infty}{\sim} \frac{H_{\mathcal{P}}(k)}{\Gamma(k^2+1)}(\log t)^{k^2},\ \ H_{\mathcal{P}}(k)=\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right)^{k^2}{}_2\mathrm{F}_1\left(k,k,1,\frac{1}{p}\right).$$

What c_k should be for general k remained mysterious until the idea that such macroscopic statistics may be related to the corresponding ones for random matrices [30]. Searching the limits and deepness of this connection is maybe, more than the direct number-theoretic consequences, the main motivation for the study of these particular statistics.

3.2 The moments conjecture

The general equivalent for the ζ moments, proposed by Keating and Snaith, takes the following form. It coincides with all previous results and conjectures, corresponding to $k=1,2,3,4$. Note the difficulty to test it numerically, because of the $\log t$ dependence. However, as we will see later in this section, a complete expansion in terms of powers of $\log t$ was also proposed, which completely agrees with numerical experiments, giving strong support for the following asymptotics.

Conjecture 3.1. *For every $k\in\mathbb{N}^*$*

$$I_k(t) \underset{t\to\infty}{\sim} H_{\mathrm{Mat}}(k)H_{\mathcal{P}}(k)(\log t)^{k^2}$$

with the notation

$$H_{\mathcal{P}}(k)=\prod_{p\in\mathcal{P}}\left(1-\frac{1}{p}\right)^{k^2}{}_2F_1\left(k,k,1,\frac{1}{p}\right)$$

for the previously mentioned arithmetic factor, and the matrix factor

$$H_{\mathrm{Mat}}(k) = \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}. \tag{30}$$

Note that the above term makes sense for imaginary k (it can be expressed by means of the Barnes G-function) so more generally this conjecture may be stated for any $\Re k \geq -1/2$.

Let us outline a few of key steps involved in understanding the origins of this conjecture. First suppose that $\sigma > 1$. Then the absolute convergence of the Euler product and the linear independence of the $\log p$'s ($p \in \mathcal{P}$) over $\mathbb{Q}$ make it possible to show that

$$\frac{1}{t}\int_0^t \mathrm{d}s \left|\zeta\left(\sigma + \mathrm{i}s\right)\right|^{2k} \underset{t\to\infty}{\sim} \prod_{p\in\mathcal{P}} \frac{1}{t}\int_0^t \frac{\mathrm{d}s}{\left|1-\frac{1}{p^s}\right|^{2k}} \underset{t\to\infty}{\longrightarrow} \prod_{p\in\mathcal{P}} {}_2F_1\left(k,k,1,\frac{1}{p^{2\sigma}}\right). \tag{31}$$

This asymptotic independence of the factors corresponding to distinct primes gives the intuition underpinning part of the arithmetic factor. Note that this equivalent of the k-th moment is guessed to hold also for $1/2 < \sigma \leq 1$, which would imply the Lindelöf hypothesis (see Titchmarsh [48]). Moreover, the factor $(1-1/p)^{k^2}$ in $H_{\mathcal{P}}(k)$ can be interpreted as a compensator to allow the RHS in (31) to converge on $\sigma = 1/2$.

In another direction, when looking at the Dirichlet series instead of the Euler product, the Riemann zeta function on the critical axis ($\Re(s) = 1/2, \Im(s) > 0$) is the (not absolutely) convergent limit of the partial sums

$$\zeta_n(s) = \sum_{k=1}^{n} \frac{1}{k^s}.$$

Conrey and Gamburd [17] showed that

$$\lim_{n\to\infty}\lim_{t\to\infty} \frac{1}{t(\log n)^{k^2}} \int_0^t \left|\zeta_n\left(\frac{1}{2}+\mathrm{i}t\right)\right|^{2k} \mathrm{d}t = H_{\mathrm{Sq}}(k) H_{\mathcal{P}}(k),$$

where $H_{\mathrm{Sq}}(k)$ is a factor distinct from $H_{\mathrm{Mat}}(k)$ and linked to counting magic squares. So the arithmetic factor appears when considering the moments of the partial sums, and inverting the above limits conjecturally changes $H_{\mathrm{Sq}}(k)$ to a different factor $H_{\mathrm{Mat}}(k)$.

The matrix factor, which is consistent with numerical experiments, comes from the idea [30] (enforced by Montgomery's theorem) that a good approximation for the zeta function is the determinant of a unitary matrix. Thanks to Selberg's integral[14], the Mellin-Fourier for the determinant of a $n\times n$ random unitary matrix ($Z_n(u,\phi) = \det(\mathrm{Id} - e^{-\mathrm{i}\phi}u)$) with respect to the Haar measure $\mu_{\mathrm{U}(n)}$ is

$$P_n(s,t) = \mathbb{E}_{\mu_{\mathrm{U}(n)}}\left(|Z_n(u,\phi)|^t e^{\mathrm{i}s \arg Z_n(u,\phi)}\right) = \prod_{j=1}^{n} \frac{\Gamma(j)\Gamma(t+j)}{\Gamma(j+\frac{t+s}{2})\Gamma(j+\frac{t-s}{2})}. \tag{32}$$

[14] More about the Selberg integrals and its numerous applications can be found in [22]

The closed form (32) implies in particular

$$\mathbb{E}_{\mu_{U(n)}}\left(|Z_n(u,\phi)|^{2k}\right) \underset{n\to\infty}{\sim} H_{\mathrm{Mat}}(k) n^{k^2}.$$

This leads one to introduce $H_{\mathrm{Mat}}(k)$ in the conjectured asymptotics of $I_k(T)$. This matrix factor is supposed to be universal : it should for example appear in the asymptotic moments of Dirichlet L-functions.

However, these explanations are not sufficient to understand clearly how these arithmetic and matrix factors must be combined to get the Keating-Snaith conjecture. A clarification of this point is the purpose of the three following paragraphs.

The hybrid model. Gonek, Hughes and Keating [23] gave an explanation for the moments conjecture based on a particular factorization of the zeta function.

Let $s=\sigma+\mathrm{i}t$ with $\sigma\geq 0$ and x a real parameter. Let $u(x)$ be a nonnegative $\mathscr{C}^\infty$ function of mass 1, supported on $[e^{1-1/x},e]$, and set $U(z)=\int_0^\infty u(x)E_1(z\log x)\mathrm{d}x$, where $E_1(z)$ is the exponential integral $\int_z^\infty (e^{-w}/w)\ \mathrm{d}w$. Let also

$$P_x(s)=\exp\left(\sum_{n\leq x}\frac{\Lambda(n)}{n^s\log n}\right)$$

where Λ is Van Mangoldt's function ($\Lambda(n)=\log p$ if n is an integral power of a prime p, 0 otherwise), and

$$Z_x(s)=\exp\left(-\sum_{\rho_n}U\left((s-\rho_n)\log x\right)\right)$$

where $(\rho_n, n\geq 0)$ are the non-trivial ζ zeros. Then unconditionally, for any given integer m,

$$\zeta(s)=P_x(s)Z_x(s)\left(1+\mathrm{O}\left(\frac{x^{m+2}}{(|s|\log x)^m}\right)+\mathrm{O}\left(x^{-\sigma}\log x\right)\right),$$

where the constants in front of the O only depend on the function u and m : this is a hybrid formula for ζ, with both a Euler and Hadamard product. The P_x term corresponds to the *arithmetic* factor of the moments conjecture, while the Z_x term corresponds to the *matrix* factor. More precisely, this decomposition suggests a proof for Conjecture 3.1 along the following steps.

First, for a value of the parameter x chosen such that $x=O(\log(t)^{2-\varepsilon})$, the following splitting conjecture states that the moments of zeta are well approximated by the product of the moments of P_x and Z_x (they are sufficiently *independent*) :

$$\frac{1}{t}\int_0^t \mathrm{d}s\left|\zeta\left(\frac12+\mathrm{i}s\right)\right|^{2k} \underset{t\to\infty}{\sim} \left(\frac{1}{t}\int_0^t \mathrm{d}s\left|P_x\left(\frac12+\mathrm{i}s\right)\right|^{2k}\right) \times\left(\frac{1}{t}\int_0^t \mathrm{d}s\left|Z_x\left(\frac12+\mathrm{i}s\right)\right|^{2k}\right).$$

Assuming that the above result is true, we then need to approximate the moments of P_x and Z_x. Concerning P_x [23] proves that

$$\frac{1}{t}\int_0^t \mathrm{d}s \left|P_x\left(\frac{1}{2}+\mathrm{i}s\right)\right|^{2k} = H_{\mathcal{P}}(k)\ (e^{\gamma}\log x)^{k^2}\left(1+O\left(\frac{1}{\log x}\right)\right).$$

Finally, an additional conjecture about the moments of Z_x would be the last step in the moments conjecture :

$$\frac{1}{t}\int_0^t \mathrm{d}s \left|Z_x\left(\frac{1}{2}+\mathrm{i}s\right)\right|^{2k} \underset{t\to\infty}{\sim} H_{\mathrm{Mat}}(k)\ \left(\frac{\log t}{e^{\gamma}\log x}\right)^{k^2}. \tag{33}$$

The reasoning which leads to this supposed asymptotic is the following. First of all, the function Z_x is not as complicated as it seems, because as x tends to ∞, the function u tends to the Dirac measure at point e, so $Z_x\left(\frac{1}{2}+\mathrm{i}s\right) \approx \prod_{\gamma_n}\left(\mathrm{i}(t-\gamma_n)e^{\gamma}\log x\right)$. The ordinates γ_n (where ζ vanishes) are supposed to have many statistical properties identical to those of the eigenangles of a random element of $U(n)$. In order to make an adequate choice for n, we recall that the γ_n have spacing $2\pi/\log t$ on average, whereas the eigenangles have mean gap $2\pi/n$: thus n should be chosen to be about $\log t$. Then the random matrix moments lead to conjecture (33).

Multiple Dirichlet series. In a completely different direction, Diaconu, Goldfeld and Hoffstein [19] proposed an explanation of the Conjecture 3.1 relying only on a supposed meromorphy property of the multiple Dirichlet series

$$\int_1^{\infty}\zeta(s_1+\varepsilon_1\mathrm{i}t)\dots\zeta(s_{2m}+\varepsilon_{2m}\mathrm{i}t)\left(\frac{2\pi e}{t}\right)^{k\mathrm{i}t} t^{-w}\mathrm{d}t,$$

with $w, s_k\in\mathbb{C}$, $\varepsilon_k=\pm1$ $(1\leq k\leq 2m)$. They make no use of any analogy with random matrices to predict the moments of ζ, and recover the Keating-Snaith conjecture. Important tools in their method are a group of approximate functional equations for such multiple Dirichlet series and a Tauberian theorem to connect the asymptotics as $w\to 1^+$ and the moments $\int_1^t$.

An intriguing question is whether their method applies or not to predict the joint moments of ζ,

$$\frac{1}{t}\int_1^t \mathrm{d}t\prod_{j=1}^{\ell}\left|\zeta\left(\frac{1}{2}+\mathrm{i}(t+s_j)\right)\right|^{2k_j}$$

with $k_j\in\mathbb{N}^*$, $(1\leq j\leq \ell)$, the s_j's being distinct elements in $\mathbb{R}$. If such a conjecture could be stated, independently of any considerations about random matrices, this would be an accurate test for the correspondence between random matrices and L-functions. For such a conjecture, one expects that it agrees with the analogous result on the unitary group, which is a special case of the Fisher-Hartwig asymptotics of

Toeplitz determinants first proven by Widom [49] :

$$\begin{aligned}
&\mathbb{E}_{\mu_{U(n)}}\left(\prod_{j=1}^{\ell}\left|\det(\mathrm{Id}-e^{\mathrm{i}\phi_j}u)\right|^{2k_j}\right)\\
\underset{n\to\infty}{\sim}&\prod_{1\leq i<j\leq\ell}|e^{\mathrm{i}\phi_i}-e^{\mathrm{i}\phi_j}|^{-2k_ik_j}\prod_{j=1}^{\ell}\mathbb{E}_{\mu_{U(n)}}\left(\left|\det(\mathrm{Id}-e^{\mathrm{i}\phi_j}u)\right|^{2k_j}\right)\\
\underset{n\to\infty}{\sim}&\prod_{1\leq i<j\leq\ell}|e^{\mathrm{i}\phi_i}-e^{\mathrm{i}\phi_j}|^{-2k_ik_j}\prod_{j=1}^{\ell}H_{\mathrm{Mat}}(k_j)n^{k_j^2},
\end{aligned}$$

the ϕ_j's being distinct elements modulo 2π.

Joint moments. Strong evidence supporting Conjecture 3.1 was obtained in [15] : an extension is given to the joint moments of the Riemann zeta function with entries shifted by constants along the critical axis. In particular, this generalized moment conjecture gives a complete expansion of $I_k(t)$ in terms of powers of $\log t$ which agrees remarkably with numerical tests.

More precisely, we denote briefly $z=(z_1,\dots,z_{2k})$ and introduce the Euler product

$$a_k(z)=\prod_{p\in\mathcal{P},1\leq i,j\leq k}\left(1-\frac{1}{1+p^{z_i-z_{j+k}}}\right)\int_0^1\mathrm{d}\theta\prod_{j=1}^{k}\frac{1}{\left(1-\frac{e^{\mathrm{i}2\pi\theta}}{p^{1/2+z_j}}\right)\left(1-\frac{e^{-\mathrm{i}2\pi\theta}}{p^{1/2-z_{j+k}}}\right)},$$

and $g(z)=a_k(z)\prod_{1\leq i,j\leq k}\zeta(1+z_i-z_{j+k})$. Then we define $P_k(x,(\alpha_i)_1^{2k})$ as the integral (the path of integration being defined as small circles surrounding the poles α_i)

$$e^{-\frac{x}{2}\sum_1^k(\alpha_i-\alpha_{i+k})}\frac{(-1)^k}{(k!)^2(\mathrm{i}2\pi)^{2k}}\oint\dots\oint e^{\frac{x}{2}\sum_1^k(z_i-z_{i+k})}\frac{g(z)\Delta^2(z)}{\prod_{1\leq i,j\leq 2k}(z_i-\alpha_j)}\mathrm{d}z_1\dots\mathrm{d}z_{2k},$$

where Δ is the Vandermonde determinant $\Delta(z)=\prod_{i<j}(z_j-z_i)$. Then the complete moments conjecture from [15] is that, for any $\varepsilon>0$,

$$\begin{aligned}
\int_0^t\prod_{i=1}^{k}\zeta\left(\frac{1}{2}+\mathrm{i}s+\alpha_i\right)\prod_{j=k+1}^{2k}\zeta\left(\frac{1}{2}+\mathrm{i}s-\alpha_j\right)\mathrm{d}s&\\
=\int_0^t P_k\left(\log\frac{s}{2\pi},(\alpha_i)_1^{2k}\right)\left(1+\mathrm{O}\left(s^{-\frac{1}{2}+\varepsilon}\right)\right)\mathrm{d}s.&\quad(34)
\end{aligned}$$

One can prove that for $\alpha_1=\dots=\alpha_{2k}=0$, P_k is polynomial in x with degree k^2 with leading coefficient as expected from Conjecture 3.1. Where does the general moments conjecture come from ? A very similar formula was obtained by the same authors [14] concerning the characteristic polynomial of a random unitary matrix, noted here $Z_u(\alpha) = \det(\mathrm{Id} - e^{-\alpha} u)$:

$$\mathbb{E}_{\mu_{\mathrm{U}(n)}}\left(\prod_{i=1}^{k} Z_u(\alpha_i) \prod_{j=k+1}^{2k} Z_{u^\dagger}(-\alpha_j)\right) = e^{-\frac{n}{2}\sum_{i=1}^{k}(\alpha_i - \alpha_{k+i})} \frac{(-1)^k}{(k!)^2 (\mathrm{i}2\pi)^{2k}}$$
$$\oint \dots \oint e^{\frac{n}{2}\sum_{i=1}^{k}(z_i - z_{k+i})} \prod_{1 \le i,j \le k} \frac{1}{1 - e^{z_{k+j} - z_i}} \frac{\Delta^2(z)}{\prod_{1 \le i,j \le 2k}(z_i - \alpha_j)} \mathrm{d}z_1 \dots \mathrm{d}z_{2k}.$$

From this formal analogy between the joint moments of unitary matrices and that of the Riemann zeta function one gets strikingly accurate numerical results. For example, the following numerical data from [15] compare the conjectural moments asymptotics when $k = 3$ (writing $P_3(x)$ for $P_3(x, (0, \dots, 0))$), on an interval I

$$\int_I P_3\left(\frac{\log s}{2\pi}\right) \mathrm{d}s \tag{35}$$

with the numerical computation for

$$\int_I \left|\zeta\left(\frac{1}{2} + \mathrm{i}s\right)\right|^6 \mathrm{d}s. \tag{36}$$

Integration domain I	Full moment conjecture (35)	Numerics (36)	Ratio
$[1300000, 1350000]$	80188090542.5	80320710380.9	1.001654
$[1350000, 1400000]$	81723770322.2	80767881132.6	.988303
$[1400000, 1450000]$	83228956776.3	83782957374.3	1.006656
$[0, 2350000]$	3317437762612.4	3317496016044.9	1.000017

3.3 Gaussianity for ζ and characteristic polynomials

The explicit computation (32) of the mixed Mellin-Fourier transform of the characteristic polynomial allows one to prove the following central limit theorem [30] : as $n \to \infty$

$$\frac{\log Z_n}{\sqrt{\log n}} \xrightarrow{\text{law}} \mathcal{N}_1 + \mathrm{i}\mathcal{N}_2, \tag{37}$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are standard independent real Gaussian random variables. A similar result holds, unconditionally, for the logarithm of the Riemann zeta function. This was shown by Selberg[15] [45].

This indicates that the correspondence between random matrices and L-functions is not uniquely observable through the microscopic repulsion, but also at a macroscopic level, like in the moment conjecture.

Theorem 3.2. *Writing ω for a uniform random variable on* $(0, 1)$*,*

$$\frac{\log \zeta\left(\frac{1}{2} + \mathrm{i}\omega t\right)}{\sqrt{\frac{1}{2}\log\log t}} \xrightarrow[t \to \infty]{\text{law}} \mathcal{N}_1 + \mathrm{i}\mathcal{N}_2. \tag{38}$$

It may be of interest, to understand the mixing properties of primes, to give ideas of the proof of this central limit theorem.

[15] Note that it predates the analogous result about random matrices (37), an unusual situation.

Sketch of proof. Suppose the Euler product of ζ holds for $1/2 \leq \Re(s) \leq 1$ (this is a conjecture) : then $\log\zeta(s) = -\sum_{p\in\mathcal{P}}\log(1-p^{-s})$ can be approximated by $\sum_{p\in\mathcal{P}} p^{-s}$. Let $s = 1/2+\varepsilon_t+\mathrm{i}\omega t$ with ω uniform on $(0,1)$. As the $\log p$'s are linearly independent over $\mathbb{Q}$, the terms $\{p^{-\mathrm{i}\omega t} \mid p\in\mathcal{P}\}$ can be viewed as independent uniform random variables on the unit circle as $t\to\infty$, hence it is a natural thought that a central limit theorem might hold for $\log\zeta(s)$.

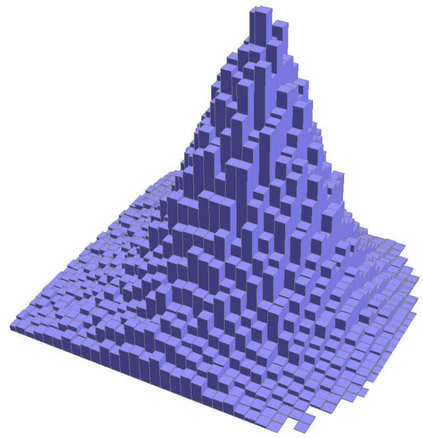


Fig. 8 – Histogram of 10^5 values of $\log\zeta$ around 10^6.

The crucial point to get such arithmetical central limit theorems is the approximation by sufficiently short Dirichlet series. More precisely, the explicit formula for ζ'/ζ, by Landau, gives such an approximation ($x>1$, s distinct from 1, the zeroes ρ and $-2n$, $n\in\mathbb{N}$) :

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n\leq x}\frac{\Lambda(n)}{n^s} + \frac{x^{1-s}}{1-s} - \sum_{\rho}\frac{x^{\rho-s}}{\rho-s} + \sum_{n=1}^{\infty}\frac{x^{-2n-s}}{2n+s},$$

from which we get an approximate formula for $\log\zeta(s)$ by integration. However, the sum over the zeros is not absolutely convergent, hence this formula is not sufficient. Selberg found a slight change in the above formula, that makes a great difference because all infinite sums are now absolutely convergent : under the above hypotheses, if

$$\Lambda_x(n) = \begin{cases} \Lambda(n) & \text{for } 1\leq n\leq x,\\ \Lambda(n)\frac{\log\frac{x^2}{n}}{\log n} & \text{for } x\leq n\leq x^2,\end{cases}$$

then

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n\leq x^2}\frac{\Lambda_x(n)}{n^s} + \frac{x^{2(1-s)}-x^{1-s}}{(1-s)^2\log x} + \frac{1}{\log x}\sum_{\rho}\frac{x^{\rho-s}-x^{2(\rho-s)}}{(\rho-s)^2} + \frac{1}{\log x}\sum_{n=1}^{\infty}\frac{x^{-2n-s}-x^{-2(2n+s)}}{(2n+s)^2}.$$

Assuming the Riemann hypothesis, the above formulas give a simple expression for $(\zeta'/\zeta)(s)$ for $\Re(s)\geq 1/2$: for $x\to\infty$, all terms in the infinite sums converge to 0 because $\Re(\rho-s)<0$. By subtle arguments, Selberg showed that, although RH is necessary for the *almost sure* coincidence between ζ'/ζ and its Dirichlet series, it is not required in order to get *a good* L^k *approximation*. In particular, for any $k\in\mathbb{N}^*$, $0<a<1$, there is a constant $c_{k,a}$ such that for any $1/2\leq\sigma\leq 1$, $t^{a/k}\leq x\leq t^{1/k}$,

$$\frac{1}{t}\int_1^t\left|\log\zeta(\sigma+\mathrm{i}s) - \sum_{p\leq x}\frac{p^{-\mathrm{i}s}}{p^{\sigma}}\right|^{2k}\mathrm{d}s \leq c_{k,a}.$$

In the following, we only need the case $k = 1$ in the above formula : with ω uniform on $(0,1)$, $\log\zeta\left(\frac{1}{2}+\mathrm{i}\omega t\right)-\sum_{p\le t}\frac{p^{-\mathrm{i}\omega t}}{\sqrt{p}}$ is bounded in L^2, and after normalization by $\frac{1}{\sqrt{\log\log t}}$, it converges in probability to 0. Hence, the central limit theorem for $\log\zeta$ is equivalent to

$$\frac{1}{\sqrt{\log\log t}}\sum_{p\le t}\frac{p^{-\mathrm{i}\omega t}}{\sqrt{p}}\xrightarrow[t\to\infty]{\text{law}}\mathcal{N}_1+\mathrm{i}\mathcal{N}_2.$$

The proof of the above result proceeds in two steps.

Firstly, the length of the Dirichlet series can still be decreased, thanks to the Montgomery-Vaughan inequality, Theorem 1.9. From the properties of linear independence of primes, there is a constant $c > 0$ independent of p with $\min_{p'\neq p}|\log p-\log p'|>\frac{c}{p}$, so for any $m_t < t$

$$\frac{1}{t}\int_0^t\left|\sum_{m_t<p<t}\frac{p^{-\mathrm{i}s}}{\sqrt{p\log\log t}}\right|^2\mathrm{d}s\le\sum_{m_t<p<t}\frac{1}{p\log\log t}\left(1+3\pi c\frac{p}{t}\right).$$

By the prime number theorem this goes to 0 provided that, as $t\to\infty$, $\frac{\log\log m_t}{\log\log t}\to 1$. If the above condition is satisfied, we therefore just need to prove

$$\sum_{p\le m_t}\frac{e^{\mathrm{i}\omega t}}{\sqrt{p\log\log t}}\xrightarrow{\text{law}}\mathcal{N}_1+\mathrm{i}\mathcal{N}_2. \tag{39}$$

Secondly, the classical central limit theorem states that

$$\sum_{p\le m_t}\frac{e^{\mathrm{i}\omega_p}}{\sqrt{p\log\log t}}\xrightarrow{\text{law}}\mathcal{N}_1+\mathrm{i}\mathcal{N}_2, \tag{40}$$

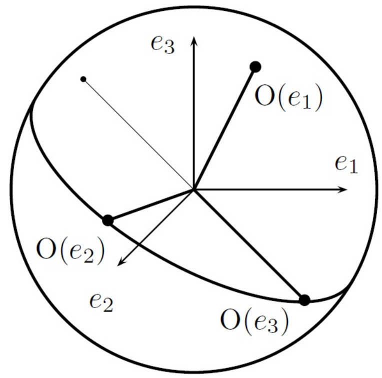
when the ω_p's are independent uniform random variables on $(0,2\pi)$. The $\log p$'s being linearly independent over $\mathbb{Q}$, it is well known that as $t\to\infty$ any given finite number of the $p^{\mathrm{i}\omega t}$'s are asymptotically independent and uniform on the unit circle. The problem here is that the number of these random variables increases as they become independent. If this number increases sufficiently slowly ($\log m_t/\log t\to 0$), one can show that (40) implies (39). This is a result about the mixing time of T^s, the translation on the torus $\mathbb{T}^n$ with vector $s(\log p_1,\dots,\log p_n)$, where both s and n go to ∞ : the mean in time (39) is very close to the mean in space (40). The method to prove it consists in computing the moments of the time average, and show that only the diagonal terms contribute (i.e. the terms corresponding to the space average). □

A natural attitude in probability, to prove a central limit theorem such as (37), consists in identifying independent random variables. However, from the uniform (Haar) measure on $\mathrm{U}(n)$, such an identification is not straightforward.

To tackle this problem we need to understand how one can generate the Haar measure as a product of independent transformations, to deduce identities in law concerning the characteristic polynomials of random matrices [11]. Let us take the example of a uniformly distributed element of $O(3)$. It seems natural to proceed as follows :

– $O(e_1)$ is uniform on the unit sphere ;

– $O(e_2)$ is uniform on the unit circle orthogonal to $O(e_1)$;
– $O(e_3)$ is uniform on $\{O(e_1)\wedge O(e_2), -O(e_1)\wedge O(e_2)\}$.



The lines hereafter are a formalization of the above simple idea, written here for the unitary group. For any $0\leq k\leq n$, note $\mathcal{H}_k := \{u \in \mathrm{U}(n) \mid u(e_j) = e_j, 1\leq j\leq k\}$, the subgroup of $\mathrm{U}(n)$ stabilizing of the first k basis vectors, and μ_k its Haar measure (in particular $\mu_0 = \mu_{\mathrm{U}(n)}$). Moreover, let p_k be the projection $u\mapsto u(e_k)$. A sequence $(\nu_1,\dots,\nu_n)$ of probability measures on $\mathrm{U}(n)$ is said *coherent* coherent with the Haar measure $\mu_{\mathrm{U}(n)}$ if for all $1\leq k\leq n$, $\nu_k(\mathcal{H}_{k-1}) = 1$ and $p_k(\nu_k) = p_k(\mu_{k-1})$.

A result of [11] asserts that if $(\nu_1,\dots,\nu_n)$ is coherent with $\mu_{\mathrm{U}(n)}$, then one has the equality of measures

$$\mu_{\mathrm{U}(n)} = \nu_1\times\nu_2\times\cdots\times\nu_n, \tag{41}$$

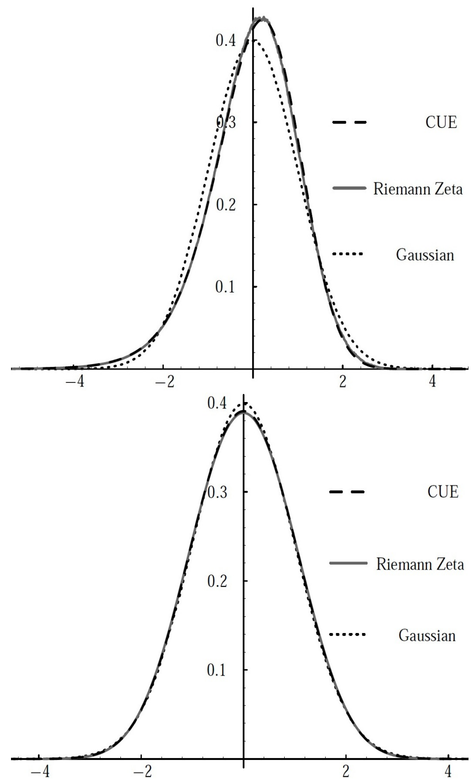
which means that to generate a uniform unitary matrix, one can proceed by multiplying n independent unitary matrices from the embedded subgroups $(\mathcal{H}_k)$, provided that their first non-trivial row is uniform on spheres of increasing size.

If one chooses reflections[16] $r_1,\dots,r_n$ for these independent transformations the projection of (41) by the determinant takes a remarkably easy form, because of the algebraic identity

$$\det\left(\mathrm{Id}-r\begin{pmatrix}1 & 0\\ 0 & u\end{pmatrix}\right) = (1-\langle e_1, r(e_1)\rangle)\det(\mathrm{Id}-u)$$

when r is a reflection. Iterating this formula in (41) yields, for $u\sim\mu_{\mathrm{U}(n)}$,

$$\det(\mathrm{Id}-u) \overset{\text{law}}{=} \prod_{k=1}^{n}\left(1-e^{\mathrm{i}\omega_k}\sqrt{\beta_{1,k-1}}\right), \tag{42}$$



where all random variables are independent, ω_k uniform on $(0,2\pi)$, and β a beta random variable[17] with indicated parameters. Indeed, the first coordinate of a unit complex vector uniformly distributed on a k-dimensional complex sphere is known to be distributed like $e^{\mathrm{i}\omega_k}\sqrt{\beta_{1,k-1}}$. The decomposition (42) also raises the (open) question of an analogue of these unexpected independent random variables in a number-theoretic context.

It also gives a direct proof of (32) as well as a derivation of the central limit theorem (37), by a simple central limit theorem for sums of independent random variables. Moreover, this decomposition yields a speed of convergence, by Berry-Esseen type theorems, and that this convergence rate is better for the imaginary part than for the real one (because $\Im\log\left(1-e^{\mathrm{i}\omega_k}\sqrt{\beta_{1,k-1}}\right)$ has a symmetric density,

[16] i.e. $\mathrm{rank}(r_k-\mathrm{Id}) = 1$ or 0.

[17] $\beta_{a,b}$ is a random variable supported on $(0,1)$ with measure $\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}\mathrm{d}x$

which is not the case for $\Re \log\left(1 - e^{\mathrm{i}\omega_k}\sqrt{\beta_{1,k-1}}\right)$. This can be compared to these histograms of $\log\zeta$ values, around height 10^{20}, from [30] and based on numerical data in [38] They illustrate the Selberg limit theorem (38), by comparing $\Re(\log\zeta)$ (up) and $\Im(\log\zeta)$ (down) with the Gaussian distribution and the density of $\log\det(\mathrm{Id} - u)$ where $u \sim \mu_{\mathrm{U}(42)}$ (the Circular Unitary Ensemble, CUE). The dimension and height are chosen to satisfy $n \approx \log t$. This shows a better agreement between U(42) and ζ statistics than with the Gaussian, and a better convergence speed for $\arg\zeta$ than for $\log|\zeta|$, like in the unitary case.

Finally, the correspondence $n \leftrightarrow \log t$ holds not only for the matrix dimension compared to the height along the critical axis, but also for small shifts away from the unit circle and the critical axis. For example, the same techniques as above yield the following phase transition in the normalization, from a constant if sufficiently close to the critical line till a lower one when going further.

If $\varepsilon_n \to 0$, $\varepsilon_n \gg 1/n$, and $u_n \sim \mu_{\mathrm{U}(n)}$, then	If $\varepsilon_t \to 0$, $\varepsilon_t \gg 1/\log t$, and ω is uniform on $(0,1)$, then
$\dfrac{\log\det(\mathrm{Id} - e^{-\varepsilon_n} u_n)}{\sqrt{-\frac{1}{2}\log\varepsilon_n}} \xrightarrow{\text{law}} \mathcal{N}_1 + \mathrm{i}\mathcal{N}_2$	$\dfrac{\log\zeta\left(\frac{1}{2} + \varepsilon_t + \mathrm{i}\omega t\right)}{\sqrt{-\frac{1}{2}\log\varepsilon_t}} \xrightarrow{\text{law}} \mathcal{N}_1 + \mathrm{i}\mathcal{N}_2$
The required normalisation becomes $\sqrt{\frac{1}{2}\log n}$ if $\varepsilon_n \ll 1/n$.	The required normalisation becomes $\sqrt{\frac{1}{2}\log\log t}$ if $\varepsilon_t \ll 1/\log t$.

3.4 Families of L-functions

Another type of statistics concerns families of L-functions, i.e. averages over a set of functions at a specific point instead of a mean along the critical axis for a given L-function. In this context, the limiting statistics are linked to distinct compact groups, including :

- U(n), the group of $n \times n$ unitary matrices, involved in many analogies as we have seen ;
- SO($2n$), the special orthogonal group, orthogonal $2n \times 2n$ matrices u with $\det(u) = 1$;
- USp($2n$), the unitary symplectic group, $2n \times 2n$ unitary matrices satisfying $u^{\mathrm{t}} J u = J$, where $J = \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}$.

A result by Katz-Sarnak [39] states that, irrespective of the choice of the three above groups, for any k the k-th consecutive spacings measures

$$\mu_k^{(u)}[a,b] = \frac{1}{n}\left|\left\{1 \le j \le n : \frac{n}{2\pi}(\theta_{j+k} - \theta_j) \in [a,b]\right\}\right|$$

are the same in the limit $n \to \infty$, where $0 \le \theta_1 \le \dots$ are the ordered eigenangles of u. These local statistics do not depend on the group in question for large dimensions. This corresponds to the universality of the GUE statistics for zeros at large height along the critical axis, for example for Dirichlet L-functions or L-functions attached to elliptic curves. Consequently, to make a statistical distinction amongst families, one needs to look at low-lying zeros, i.e. close to the symmetry point 1/2. They correspond to eigenvalues close to 1 on the unit circle.

One example of a family of L-functions is the following, attached to real quadratic Dirichlet characters. For a prime p, let $\chi_d(p) = 0$ if $p \mid d$, 1 if $p \nmid d$ and d is a square modulo p, and -1 otherwise. Then define the Dirichlet L-function for $\Re(s) > 1$ by

$$\mathrm{L}^{(\mathrm{Dir})}(s, \chi_d) = \prod_{\mathcal{P}} \frac{1}{1 - \frac{\chi_d(p)}{p^s}} = \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s},$$

where the definition of χ_d is extended to $\mathbb{N}$ and satisfies the multiplicative property $\chi_d(mn) = \chi_d(m)\chi_d(n)$. Then $\mathrm{L}^{(\mathrm{Dir})}$ can be meromorphically extended to $\mathbb{C}$, satisfies a functional equation, and its central statistics $\mathrm{L}^{(\mathrm{Dir})}(1/2, \chi_d)$ are supposedly linked to the Haar measure of the unitary symplectic group. For this group, the eigenvalues are symmetrically positioned about the real axis (like for the zeros of $\mathrm{L}^{(\mathrm{Dir})}$) and with eigenangles density proportional to

$$\prod_{1 \leq j < k \leq n} (\cos\theta_j - \cos\theta_k)^2 \prod_{j=1}^{n} \sin^2\theta_j. \tag{43}$$

The conjecture of Conrey-Farmer [13] and Keating-Snaith [31] is that, for any integer k (they extend it to real k thanks to the Barnes function),

$$\frac{1}{D^*} \sum_{|d| \leq D}^{*} \mathrm{L}(1/2, \chi_d)^k \underset{D \to \infty}{\sim} \mathcal{H}_{\mathrm{Mat}}^{(\mathrm{USp})}(k)\ \mathcal{H}_{\mathcal{P}}^{(\mathrm{Dir})}(k) \left(\frac{1}{2}\log D\right)^{\frac{k(k+1)}{2}},$$

where $\sum^*$ means that the summation is restricted to fundamental discriminants[18], D^* is the number of terms in the sum, and

$$\mathcal{H}_{\mathrm{Mat}}^{(\mathrm{USp})}(k) = \lim_{n \to \infty} \frac{1}{n^{\frac{k(k+1)}{2}}} \mathbb{E}_{\mu_{\mathrm{USp}(k)}}\left(|\det(\mathrm{Id} - u)|^k\right) = 2^{\frac{k(k+1)}{2}} \prod_{j=1}^{k} \frac{j!}{(2j)!}, \tag{44}$$

$$\mathcal{H}_{\mathcal{P}}^{(\mathrm{Dir})}(k) = \prod_{\mathcal{P}} \frac{\left(1 - \frac{1}{p}\right)^{\frac{k(k+1)}{2}}}{1 + \frac{1}{p}} \left(\frac{\left(1 - \frac{1}{\sqrt{p}}\right)^{-k} + \left(1 + \frac{1}{\sqrt{p}}\right)^{-k}}{2} + \frac{1}{p}\right).$$

The limit in (44) can be performed either by the Weyl integration formula (43) and Selberg integrals asymptotics, or by a decomposition as a product of independent random variables like (42).

Examples of L-functions families featuring orthogonal symmetry and $\mathrm{SO}(2n)$ statistics can be found in [13]. This includes for example L-functions associated with elliptic curves and twisted by Dirichlet characters. For all families, with unitary, orthogonal or symplectic type, the conjectured asymptotics of low-lying zeros are relevant with numerical calculations [41].

Citing Katz and Sarnak, *we believe that the further understanding of the source of such symmetries holds the key to finding a natural spectral interpretation of the zeros.*

[18] d is a fundamental discriminant if any decomposition $d = d_0 f^2$, with d_0 a discriminant and $f \in \mathbb{N}$, implies $f = 1$

References

[1] G.W. Anderson, A. Guionnet, O. Zeitouni, *An Introduction to Random Matrices*, Cambridge University Press, 2009.

[2] N. Argaman, F.M. Dittes, E. Doron, J.P. Keating, A. Kitaev, M. Sieber, U. Smilansky, *Correlations in the actions of periodic orbits derived from quantum chaos*, Phys. Rev. Lett. **71** (1993), 4326–4329.

[3] M.V. Berry, M. Tabor, *Level clustering in the regular spectrum*, Proc. Roy. Soc. Lond. A **356** (1977), 375–394.

[4] M.V. Berry, *Semiclassical formula for the number variance of the Riemann zeros*, Nonlinearity **1** (1988), 399–407.

[5] M.V.Berry, J.P. Keating, *The Riemann zeros and eigenvalue asymptotics*, SIAM Review **41** (1999), 236–266.

[6] E.B Bogomolny, J.P. Keating, *Random matrix theory and the Riemann zeros I : three and four-point correlations*, Nonlinearity **8** (1995), 1115–1131.

[7] E.B. Bogomolny, J.P. Keating, *Random matrix theory and the Riemann zeros II : n-point correlations*, Nonlinearity **9** (1996), 911–935.

[8] E.B. Bogomolny, J.P. Keating, *Gutzwiller's trace formula and spectral statistics : Beyond the diagonal approximation*, Phys. Rev. Lett. **77** (1996), 1472–1475.

[9] O. Bohigas, M.-J. Giannoni, C. Schmidt, *Characterization of chaoic quantum spectra and universality of level fluctuation laws*, Phys. Rev. Lett. **52** (1984), 1–4.

[10] E. Bogomolny, O. Bohigas, P. Leboeuf, A.G. Monastra, *On the spacing distribution of the Riemann zeros : corrections to the asymptotic result*, J. Phys. A : Math. Gen. **39** (2006), 10743–10754.

[11] P. Bourgade, C.P. Hughes, A. Nikeghbali, M. Yor, *The characteristic polynomial of a random unitary matrix : a probabilistic approach*, Duke Mathematical Journal **145**, 1 (2008), 45–69.

[12] A. Connes, *Trace formula in non-commutative Geometry and the zeros of the Riemann zeta function*, Selecta Math. (N.S.) **5**, no. 1 (1999), 29–106.

[13] J.B. Conrey, D.W. Farmer, *Mean values of L-functions and symmetry*, International Mathematics Research Notices, 883–908, 2000.

[14] J.B. Conrey, D.W. Farmer, J. P. Keating, M. O. Rubinstein, and N. C. Snaith, *Autocorrelation of random matrix polynomials*, Commun. Math. Phys. **237** 3 (2003), 365–395.

[15] J.B. Conrey, D.W. Farmer, J.P. Keating, M.O. Rubinstein, N.C. Snaith, *Integral moments of L-functions*, Proc. London Math. Soc. **91** (2005), 33–104.

[16] J.B. Conrey, A. Ghosh, *On mean values of the zeta function, iii*, Proceedings of the Amalfi Conference in Analytic Number Theory, Università di Salerno, 1992.

[17] J.B. Conrey, A. Gamburd, *Pseudomoments of the Riemann zeta-function and pseudomagic squares*, Journal of Number Theory **117** (2006), 263–278, Issue 2.

[18] J.B. Conrey, S. M. Gonek, *High moments of the Riemann zeta function*, Duke Math. J. **107** (2001), 577–604.

[19] A. Diaconu, D. Goldfeld, J. Hoffstein, *Multiple Dirichlet Series and Moments of Zeta and L-Functions*, Compositio Mathematica **139**, N. 3, December (2003), 297–360 (64).

[20] P. Diaconis, M. Shahshahani, *On the eigenvalues of random matrices*, Studies in applied probability, J. Appl. Probab. **31A** (1994), 49–62.

[21] L. Erdös, J. Ramirez, B. Schlein, T. Tao, V. Vu, H.T. Yau, *Bulk universality for Wigner hermitian matrices with subexponential decay*, to appear in Math. Res. Lett.

[22] P.J. Forrester, S. Ole Warnaar, *The importance of the Selberg integral*, Bull. Amer. Math. Soc. **45** (2008), 489–534.

[23] S.M. Gonek, C.P. Hughes, J.P. Keating, *A hybrid Euler-Hadamard product formula for the Riemann zeta function*, Duke Math. J. **136**, 3 (2007), 507–549.

[24] G.H. Hardy, J.E. Littlewood, *Contributions to the theory of the Riemann zeta-function and the theory of the distributions of primes*, Acta Mathematica **41** (1918).

[25] G.H. Hardy, J.E. Littlewood, *Some Problems of Partitio Numerorum III, On the Expression of a Number as a Sum of Primes*, Acta Math. **44** (1923), 1–70.

[26] D.R. Heath-Brown, *Fractional moments of the Riemann zeta-function*, J. London Math. Soc. **24** (1981), 65–78.

[27] D. Hejhal, *On the triple correlation of the zeroes of the zeta function*, IMRN **7** (1994), 293–302.

[28] R. Holowinsky, K. Soundararajan, *Mass equidistribution for Hecke eigenforms*, to appear in Annals of Mathematics.

[29] A.E. Ingham, *Mean-value theorems in the theory of the Riemann zeta-function*, Proceedings of the London Mathematical Society **27** (1926), 278–290.

[30] J.P. Keating, N.C. Snaith, *Random Matrix Theory and $\zeta(1/2+it)$*, Comm. Math. Phys. **214** (2000), 57–89.

[31] J.P. Keating, N.C. Snaith, *Random matrix theory and L-functions at $s=1/2$*, Comm. Math. Phys. **214** (2000), 91–110.

[32] J. Marklof, *Pair correlation densities of inhomogeneous quadratic forms II*, Duke mathematical journal **115**, No. 3 (2002), 409–434.

[33] M. L. Mehta, *Random matrices*, Third edition, Pure and Applied Mathematics Series **142**, Elsevier, London, 2004.

[34] H.L. Montgomery, *The pair correlation of zeros of the zeta function*, Analytic number theory (Proceedings of Symposium in Pure Mathemathics **24** (St. Louis Univ., St. Louis, Mo., 1972), American Mathematical Society (Providence, R.I., 1973), pp. 181–193.

[35] D.A. Goldston, H.L. Montgomery, *Pair correlation of zeroes and primes in short intervals*, Analytic Number Theory and Diophantine Problems, Birkhäuser, Boston, Mass. 1987, 183–203.

[36] M/C. Gutzwiller, *Periodic orbits and lassical quantization conditions*, J. Math. Phys. **12** (1971), 343–358.

[37] H.L. Montgomery, R.C. Vaughan, *Hilbert's inequality*, J. London Math. Soc. (2), **8** (1974), 73–82.

[38] A.M. Odlyzko, *On the distribution of spacings between the zeros of the zeta function*, Math. Comp. **48** (1987), 273–308.

[39] N.M. Katz, P. Sarnak, *Random Matrices*, Frobenius Eigenvalues and monodromy, American Mathematical Society Colloquium Publications, 45. American Mathematical Society, Providence, Rhode island, 1999.

[40] B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsberichte der Berliner Akademie, Gesammelte Werke, Teubner, Leipzig, 1892.

[41] M. Rubinstein, *Evidence for a spectral interpretation of the zeros of L-functions*, PhD thesis, Princeon University.

[42] Z. Rudnick, P. Sarnak, *Zeroes of principal L-functions and random matrix theory*, Duke Math. J. **81** (1996), no. 2, 269–322. A celebration of John F. Nash.

[43] A.I. Schnirelman, *Ergodic properties of eigenfunctions*, Usp. Math. Nauk **29** (1974), 181182.

[44] A. Selberg, *Uber einen Satz von A. Gelfond*, Arch. Math. Naturvid. **44** (1941), 159–171.

[45] A. Selberg, *Old and new conjectures and results about a class of Dirichlet series*, Proceedings of the Amalfi Conference on Analytic Number Theory (Maiori, 1989), 367–385, Univeristà di Salerno, Salerno, 1992.

[46] A Soshnikov, *Determinantal random point fields*, Russ. Math. Surv. 55 (5) (2000), 923–975.

[47] K. Soundararajan, *Moments of the Riemann zeta-function*, Annals of Mathematics **170** (2009), 981–993.

[48] E.C. Titchmarsh, *The Theory of the Riemann Zeta Function*, London, Oxford University Press, 1951.

[49] H. Widom, *Toeplitz determinants with singular generating functions*, Amer. J. Math. **95** (1973), 333–383.