BULK UNIVERSALITY FOR ONE-DIMENSIONAL LOG-GASES

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In this note we consider β-ensembles with real analytic potential and arbitrary inverse temperature β, and review some recent universality results for these measures, obtained in joint works with L. Erdős and H.-T. Yau. In the limit of a large number of particles, the local eigenvalues statistics in the bulk are universal: they coincide with the spacing statistics for the Gaussian β-ensembles. We also discuss the proof of the rigidity of the particles up to the optimal scale $N^{-1+\varepsilon}$.

Keywords: β-ensembles, universality, log-gas.

Consider N ordered particles with distribution given by a log-gas at inverse temperature β, with an external potential V, and a normalization such that the joint density of the particles (with respect to the Lebesgue measure on the simplex $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N$) is

$$\frac{1}{\tilde{Z}_{N,\beta}} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\frac{N}{4}\beta \sum_i V(\lambda_i)} \mathrm{d}\lambda. \tag{1}$$

The macroscopic asymptotics of these particles are not universal: the equilibrium measure ρ_V minimizes $\mu \mapsto \int V(x)\mathrm{d}\mu(x) - \frac{1}{2}\iint_{x<y} \log|x-y|\mathrm{d}\mu(x)\mathrm{d}\mu(y)$ and depends on V (this functional has a unique minimizer when $\liminf_{x\to\pm\infty} V(x)/\log|x| > 4$, which we assume from now).

Our goal in this note is to explain the following result, which states that the microscopic interactions are universal in the large N limit, depending only on the inverse temperature β. This was proved for convex V in Ref. 5 and with no convexity assumption in Ref. 6. It states that the correlation functions[a] $\rho_k^{(V,N)}$ of the point process $\chi = \sum \delta_{\lambda_i}$ converge to those of a limiting point process, $\sin_\beta$ (whose

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[a]The correlation functions of a point process χ satisfy

$$\rho_k^{(N)}(x_1,\ldots,x_k) = \lim_{\varepsilon\to 0} \varepsilon^{-k}\, \mathbb{P}(\chi(x_i, x_i+\varepsilon) = 1, 1 \leq i \leq k).$$

If χ is almost surely supported on N points, the integration property $(Nk)\rho_k^{(N)}(x_1,\ldots,x_k) = \int_{\mathbb{R}} \rho_k^{(N)}(x_1,\ldots,x_{k+1})\mathrm{d}x_{k+1}$ also holds. We refer to Ref. 17 for a rigorous definition..

definition we review in the next section) irrespectively of the external potential V.

Theorem. *For fixed $\beta > 0$, consider the measure (1). If V is real analytic, $\inf_{\mathbb{R}} V'' > -\infty$ and the support of ρ_V is connected, then the bulk local interactions are described in terms of the $\sin_\beta$ process (defined at the end of Section 1): for any x in the support of ρ_V and $\epsilon > N^{-1+\delta}$,*

$$\frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} \frac{1}{\rho_V(x')^k}\int_{\mathbb{R}^k} f(u_1,\dots,u_k)\rho_k^{(V,N)}\left(x'+\frac{u_1}{N\rho_V(x)},\dots,x'+\frac{u_k}{N\rho_V(x)}\right)\mathrm{d}x'\mathrm{d}u \\ \underset{N\to\infty}{\longrightarrow} \int_{\mathbb{R}^k} f(u_1,\dots,u_k)\rho^{(k)}_{\sin_\beta}(u_1,\dots,u_k)\mathrm{d}u, \quad (2)$$

where we abbreviate $\mathrm{d}u = \mathrm{d}u_1\dots\mathrm{d}u_k$.

In Section 1, we review the main motivations for this result, from Random Matrix Theory. Section 2 summarizes some known universality results for randpm matrices. Sections 3 and 4 give elements towards the proof of the above theorem.

1. The Invariant Ensembles and Coulomb gases

Wigner's universality surmise about random spectra states that, although the macroscopic statistics (like the equilibrium measure) depend on the system, the microscopic statistics are independent of all details except the symmetries. The core of random matrix theory was therefore summarized in this way: *the Hamiltonian which governs the behavior of a complicated system is a random symmetric matrix with no particular properties except for its symmetric nature* (Wigner, Ref. 29, 1961). Following Wigner, Dyson, Gaudin and Mehta, Random Matrix Theory was immediately appreciated also for its mathematical interest: *not only does it have immediate usefulness and validity for real physical systems but, from the mathematical point of view, it has given rise to profound results and makes use of the deepest theorems of analysis* (Lieb and Mattis, Ref. 20, 1966). In this section, we review the limiting point processes, which are supposedly universal. Their descriptions was obtained by analyzing specific integrable matrix models.

To make the problem simpler, the matrix models initially considered are the so-called Gaussian ensembles, measures on the set of $N \times N$ matrices presenting many invariances, and therefore computable paradigms for Dyson's orthogonal, unitary or symplectic class. For example the Gaussian Unitary Ensemble is uniquely defined (up to a scaling) on the set of Hermitian matrices by the following two properties:

(a) Invariance by unitary conjugacy $H \mapsto U^*HU$, $U \in \mathrm{U}(N)$.
(b) Independence of the $H_{i,j}$'s, $i \le j$.

The entries are complex Gaussian random variables, and the joint density for

the eigenvalues is

$$\frac{1}{Z_N^{(\beta)}} \prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\beta \frac{N}{4} \sum_i \lambda_i^2} \tag{3}$$

after proper normalization, where $\beta = 2$. For matrix ensembles with the independence condition (b) still satisfied, but symmetric with invariance by orthogonal conjugacy (resp. quaternionic self-dual with invariance by symplectic conjugacy), the entries are still Gaussian and the density formula (3) for the spectrum still holds with $\beta = 1$ (resp. $\beta = 4$). They are called Gaussian Orthogonal Ensemble (resp. Gaussian Symplectic Ensemble).

The macroscopic asymptotics of the above spectral measure are described by Wigner's semicircle law: the empirical spectral measure $\frac{1}{N}\sum_{k=1}^N \delta_{\lambda_k}$ converges in probability to the distribution

$$\rho_{\mathrm{sc}}(x)\mathrm{d}x = \frac{1}{2\pi}\sqrt{(4-x^2)_+}\mathrm{d}x$$

as $N \to \infty$. Wigner proved it by the method of moments: for any integer k, $N^{-1}\operatorname{Tr}(H^k)$ converges to the k-th moment of the semicircle law. His method applies irrespectively of the distribution of the independent entries (provided that they are centered and have some properly normalized second moment), allowing him to prove the first universality result, at the macroscopic level. Wigner's motivation was to understand the microscopic interactions between energy levels. In particular, he predicted that the gap probability for eigenvalues in the bulk decays with a Gaussian tail, $\mathbb{P}(\lambda_{i+1} - \lambda_i > s/N) \approx \exp(-cs^2)$, contrasting with the exponential law observed if the eigenvalues λ_k's were independent. This difficult analysis of the gaps at the microscopic scale was performed by Gaudin, Mehta and Dyson, for the three Gaussian ensembles. For example, for $\beta = 2$, by an original argument by Gaudin , all the correlation functions up to $\rho_1^{(N)}$ satisfy the following microscopic asymptotics: if $x \in (-2, 2)$ then

$$\rho_k^{(N)}\left(x + \frac{u_1}{N\rho_{sc}(x)}, \dots, x + \frac{u_k}{N\rho_{sc}(x)}\right) \underset{N\to\infty}{\longrightarrow} \det_{k\times k} K(u_i - u_j), \tag{4}$$

where $K(u) = \frac{\sin(\pi u)}{\pi u}$. This was proved by observing that the eigenvalues distribution is a determinantal point process with explicit kernel involving the Hermite polynomials, whose asymptotics are performed by the Laplace method (see e.g. Ref. 1 for more precisions). In the cases of orthogonal and symplectic invariant ensembles, such asymptotics exist, involving Pfaffians instead of determinants.

Wigner and Dyson's idea is that these simple models are paradigms for the spectrum of general disordered systems. Supposedly, if the Hamiltonian dynamics present time reversal invariance, the GOE gives the limiting local statistics of the spectrum, while the limiting GUE statistics (4) are relevant if there is no time reversal symmetry and the GSE for time-reversal symmetry but no rotational symmetry. A striking example, extensively tested numerically, asserts that the (quantum) energy levels of (classic) chaotic billiards satisfy a repulsion of GOE type, see Ref. 4.

As a distribution of points in $\mathbb{R}$, the measure (3) describes particles confined by a quadratic external potential and a Coulomb interaction with inverse temperature β. We will refer to this measure as the Gausian β-ensemble. A natural question is whether a simple description of the microscopic limit exists for arbitrary $\beta > 0$. Valko and Virág gave such a description, not in terms of the limiting correlation functions but through an explicit construction of the limiting point process. An important ingredient for their proof in that the measure (3) is the distribution of the spectrum of a tridiagonal matrix, a description given by Trotter[27] in the cases $\beta = 1$, 2 and 4, and generalized to any β by Dumitriu and Edelman.[12] More precisely, consider the symmetric matrix

$$M(N) = \frac{1}{\sqrt{\beta N}} \begin{pmatrix} \mathcal{N}_0 & \chi_{(N-1)\beta} & 0 & 0 \\ \chi_{(N-1)\beta} & \mathcal{N}_1 & \chi_{(N-2)\beta} & 0 \\ 0 & \chi_{(N-2)\beta} & \mathcal{N}_2 & \chi_{(N-3)\beta} \\ 0 & 0 & \ddots & \ddots \end{pmatrix},$$

the $\mathcal{N}_k$'s being standard Gaussian random variables, and the χ_λ's independent with distribution $c_\lambda x^{\lambda-1} e^{-x^2/2} \mathrm{d}x$, all the upper triangle variables being independent. Then the eigenvalues of $M(N)$ have distribution (3).

This provides a description of the microscopic interactions in terms of a family of stochastic differential equation, by a discrete versions of the phase functions in the Sturm-Liouville theory. More precisely, let $f(t) = \frac{\beta}{4} e^{-\frac{\beta}{4}t}$, and consider the coupled solutions α_λ of the stochastic differential equation

$$\mathrm{d}\alpha_\lambda = \lambda f \mathrm{d}t + \Re\left((e^{-\mathrm{i}\alpha_\lambda} - 1)\mathrm{d}Z\right),$$

where Z is a bidimensional Brownian motion. Then $\frac{\alpha_\lambda}{2\pi}$ converges to some $\alpha_\lambda(\infty) \in \mathbb{Z}$, an increasing function of λ, so $N(\lambda) = \alpha_\lambda(\infty)$ is the repartition function of a point process, called $\sin_\beta$. Valkó and Virág introduced the above definition and proved that, for any $x \in (-2, 2)$, the following weak convergence holds:

$$N\rho_{sc}(x)\left(\sum_k \delta_{\lambda_k} - x\right) \to \sin_\beta .$$

For $\beta \in \{1, 2, 4\}$, this provides an alternative description of the limiting point process for the three symmetry classes, and this also gives extension to any $\beta > 0$. The limiting objects being introduced, we now review which models lie in their universality classes.

2. Universality results

Microscopic universality for random matrices is now well understood by either relaxing the invariance assumption (a) or the independence of the entries (b) in the definition of the Gaussian ensembles.

For self-dual matrices with independent entries, not necessarily Gaussian, spectacular progress occurred in the past five years, including the series of papers (see

e.g. Ref. 13,14) by Erdős, Knowles, Schlein, Yau, Yin which led for example to the following universality result. In this statement, the notation $\rho_k^{\mathrm{Wig}(N)}$ describes the k-th correlation function for the point processes $\sum_{k=1}^N \delta_{\lambda_k^{(N)}}$ where the $\lambda_k^{(N)}$'s are the eigenvalues of $W^{(N)}$, an Hermitian Wigner matrix: its entries are of type $W_{ij}^{(N)} = \frac{1}{\sqrt{2N}}(X_{ij} + \mathrm{i}Y_{ij}), 1 \le i < j \le N$, $W_{ii}^{(N)} = \frac{1}{\sqrt{N}} X_{ii}, 1 \le i \le N$, all variables being independent, centered, with variance 1 and finite moment of order $4 + \varepsilon$.

Theorem 2.1. *Under the above hypothesis, for any $x \in (-2, 2)$ and $\varepsilon > N^{-1+\delta}$,*

$$\frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} \frac{1}{\rho_{sc}(x')^k}\int_{\mathbb{R}^k} f(u_1,\dots,u_k)\rho_k^{\mathrm{Wig}(N)}\left(x' + \frac{u_1}{N\rho_{sc}(x)},\dots,x' + \frac{u_k}{N\rho_{sc}(x)}\right)\mathrm{d}x'\mathrm{d}u$$

converges to $\int_{\mathbb{R}^k} f(u_1,\dots,u_k)\det_{k\times k} K(u_i - u_j)\mathrm{d}u$.

The method to prove this theorem is very general and was applied to many other random matrix models with independent entries (covariance matrices, Erdős-Rényi graphs and generalized Wigner matrices, i.e. when the variance of the entries is allowed to vary). It also applies to prove universality for symmetric (resp. self-dual quaternionic matrices), with limiting correlation function corresponding to $\beta = 1$ (resp. $\beta = 4$). Another approach was developed in Ref. 26, proving universality for the Hermitian class, and under the extra assumption (in the symmetric and symplectic classes) that the first four moments of the entries need to match the Gaussian ones.

Important ideas for the proof of Theorem 2.1 include:

(i) A strong rigidity estimate on the ordered eigenvalues $\lambda_1^{(N)} < \cdots < \lambda_N^{(N)}$: in the bulk these eigenvalues are concentrated around their typical location up to scale $N^{-1+\varepsilon}$ for any $\varepsilon > 0$. This was obtained by a very precise analysis of the self-consistent equation, which relates the Stieltjes transforms of $W^{(N)}$ and its minors.

(ii) An understanding of the local relaxation time for Dyson's Brownian motion, which allows to interpolate between Wigner and Gaussian ensembles. Thanks to the rigidity estimate from the previous step, this time for the local relaxation of the eigenvalues dynamics is shown to be $\mathrm{O}(N^{-1+\varepsilon})$ for arbitrary small $\varepsilon > 0$.

(iii) A Green function comparison theorem, to remove the difference of eigenvalues statistics between $W^{(N)}$ and its very small perturbation, $W^{(N)} + (N^{-1+\varepsilon})^{1/2}H^{(N)}$ ($H^{(N)}$ being an element from the GUE).

Other spectacular progress about universality occurred, in the past 20 years, when keeping the invariance property but relaxing the independence. More precisely, the orthogonal (resp. unitary, symplectic) invariant ensembles are the probability measures with density $\frac{1}{Z_{N,\beta}} e^{-\frac{N}{4}\operatorname{Tr} V(M)}$ with respect to the Lebesgue measure on the set of $N \times N$ symmetric (resp. Hermitian, self-dual quaternionic) matrices. The Gaussian ensembles correspond to a quadratic V. Thanks to the conjugacy invariance of this model, the distribution of the eigenvalues is explicitly computable,

and exactly coincides with (1), in the special case $\beta = 1$ (resp. $\beta = 2, 4$). As we already mentioned, the eigenvalues statistics are not universal at the macroscopic level, but at the microscopic scale they are, as shown in the following important result.

Theorem 2.2. *Let $\beta = 2$. If V is real analytic and x is in the bulk of the limiting spectral measure $\rho_V(x)\mathrm{d}x$, then $\rho_k^{(V,N)}\left(x + \frac{u_1}{N\rho_V(x)}, \dots, x + \frac{u_k}{N\rho_V(x)}\right)$ converges as $N \to \infty$ to $\det_{k\times k} K(u_i - u_j)$. Similar results hold for the orthogonal and symplectic invariant ensembles, the limit being explicit in terms of Pfaffians.*

Note that (2) gives a similar result for arbitrary $\beta > 0$, but in a less precise way than the above theorem, due to the ε-averaging.

Many important contributions towards Theorem 2.2 included Deift, Kriecherbauer, McLaughlin, Venakides, Zhou[9] , Bleher, Its[3] , Deift, Gioev[8] , Pastur, Schcherbina[23,24] , Scherbina[25] , Lubinsky[21] . In particular, in this last work, the analyticity condition was removed when $\beta = 2$, but integrability is still essential in the proof. Indeed, it relies on the fact that for fixed N, the distribution of the spectrum is a determinantal or Pfaffian point process, with an explicit kernel in terms of the orthogonal polynomials with respect to $e^{-V(x)}\mathrm{d}x$ (Dyson, Mehta[22]). Then it was proved that these orthogonal polynomials can be evaluated from the solution of a Riemann-Hilbert problem (Fokas, Its, Kitaev[16]). A steepest descent analysis of this Riemann-Hilbert problem was performed in a seminal work by Deift and Zhou[11] .

The motivation for the result (2) is both the microscopic universality for one dimensional log-gases, and an understanding of the universality for invariant ensembles, by arguments close to statistical physics. In the next sections, we aim at explaining the two main steps for a proof of (2):

- Uniqueness of log gases with suitable boundary conditions (Section 3).
- Rigidity of the particles location (Section 4).

The proof therefore shares the same philosophy as the one of Theorem 2.1, but the techniques differ in many points: steps (1) and (3) cannot have an analogue because our β-ensembles lack any matrix model, and for step (2) there are no obvious dynamics for extrapolation, the idea of using the Dyson Brownian motion will be applied to some conditional measures, that we explain in the next section.

3. The local equilibrium measure

We rename the ordered particles $\lambda_1 < \cdots < \lambda_N$ as $y_1 < \cdots < y_L < x_{L+1} < \cdots < x_{L+K} < y_{L+K+1} < \cdots < y_N$, where $\mathbf{x} = (x_{L+1}, \dots x_{L+K})$ are the internal points and $\mathbf{y} = (y_1, \dots, y_L, y_{L+K+1}, \dots, y_N)$ the external points. We also denote

$I = [\![L+1, L+K]\!]$. The conditional measure of $\mathbf{x}$ knowing $\mathbf{y}$ is

$$\mu_{\mathbf{y}}(\mathbf{x}) = \frac{\mu(\mathbf{y},\mathbf{x})}{\int \mu(\mathbf{y},\mathbf{x})\mathrm{d}\mathbf{x}} = \frac{1}{\mathcal{Z}_{\mathbf{y}}} e^{-\beta N \mathcal{H}_{\mathbf{y}}(\mathbf{x})},$$
$$\mathcal{H}_{\mathbf{y}}(\mathbf{x}) = \frac{1}{4}\sum_{i\in I} V_{\mathbf{y}}(x_i) - \frac{1}{N}\sum_{i,j\in I, i<j} \log|x_j - x_i|,$$
$$V_{\mathbf{y}}(x) = V(x) - \frac{1}{N}\sum_{j\notin I} \log|x - y_j|.$$

For a small parameter $\kappa > 0$, let

$$\mathcal{G}_\delta = \{\mathbf{y} \in \mathbb{R}^{N-K} \mid \forall j \in [\![\kappa N, L]\!] \cup [\![L+K+1, (1-\kappa)N]\!], |y_j - \gamma_j| \le \delta\}$$

denote the set of good external configurations (some weak form of rigidity is still needed for eigenvalues on the edge, but we omit this non-essential point here).

The above definitions are made for two distinct external potentials V and $\tilde{V}$, the external points being $\mathbf{y}$, $\tilde{\mathbf{y}}$, the equilibrium measures ρ_V and $\rho_{\tilde{V}}$, and the conditional measures $\mu_{\mathbf{y}}$ and $\tilde{\mu}_{\tilde{\mathbf{y}}}$. To compare these two measures, after a proper translation and dilatation we can assume

$$[y_L, y_{L+K-1}] = [\tilde{y}_L, \tilde{y}_{L+K-1}].$$

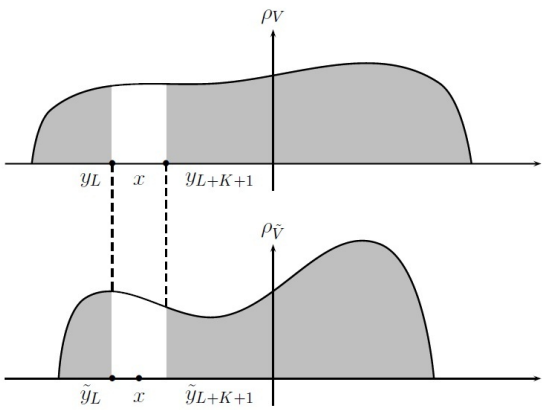

Fig. 1. Matching the local densities of ρ and $\tilde{\rho}$.

Conditionally to good boundary conditions, the local statistics are universal, as quantified in the following result (see Proposition 4.2 and Theorem 4.4 in Ref. 5).

Proposition 3.1. *Let $0 < \varphi < \frac{1}{38}$, $K = N^{\frac{39}{2}\varphi}$ and $\delta = N^{\varphi-1}$. Then for any $\mathbf{y} \in \mathcal{G}_\delta$, and smooth compactly supported test function G, $\tilde{\mathbf{y}} \in \tilde{\mathcal{G}}_\delta$, we have*

$$\left(\mathbb{E}_{\mu_{\mathbf{y}}} - \mathbb{E}_{\tilde{\mu}_{\tilde{\mathbf{y}}}}\right)\left(\frac{1}{K}\sum_I f(N(x_i - x_{i+1}))\right) \longrightarrow 0. \tag{5}$$

To get some intuition for this result, imagine that the Hessian of $\mathcal{H}_{\mathbf{y}}$ is convex, with lower bound $\nabla^2\mathcal{H}_{\mathbf{y}} \ge \tau^{-1}$. Then the Bakry-Émery criterion[2] states that $\mu_{\mathbf{y}}$ satisfies a logarithmic Sobolev inequality with constant τ. More precisely, denote $q = \frac{\mathrm{d}\tilde{\mu}_{\tilde{\mathbf{y}}}}{\mathrm{d}\mu_{\mathbf{y}}}$, $\mathrm{D}_\mu(f) = \frac{1}{N}\int |\nabla f|^2 \mathrm{d}\mu$ the Dirichlet form, and for a probability density f, $\mathrm{S}_\mu(f) = \int f \log f \mathrm{d}\mu$ the entropy. By the inequality between total variation and entropy, and then the logarithmic Sobolev inequality, for some universal $c_1, c_2 > 0$, (5) can be bounded by

$$\int |q-1| \mathrm{d}\mu_{\mathbf{y}} \le c_1\sqrt{\mathrm{S}_{\mu_{\mathbf{y}}}(q)} \le c_2\sqrt{\tau \mathrm{D}_{\mu_{\mathbf{y}}}(\sqrt{q})}.$$

However, this inequality is not sufficient: an explicit computation shows that, on the set of good configurations $\mathbf{y}$ and $\tilde{\mathbf{y}}$:

- the convexity is of order at most $\tau^{-1} = N/K$: it mostly comes from the interaction between the x_i's and y_j's, and evaluated at $x = x_{L+\lfloor K/2 \rfloor}$, $\frac{1}{N}\sum_j \partial_{xx}(-\log|x-y_j|) \approx \frac{1}{N}\sum_{k\geq K/2} \frac{1}{(k/N)^2} \approx N/K$. Defining the typical location of y_k by γ_k where $\int_{-\infty}^{\gamma_k} \rho_V(s)\mathrm{d}s = \frac{k}{N}$, the previous approximations are justified if y_k is concentrated around γ_k up to some error of order K/N.
- the Dirichlet form is of order at least N (up to logarithmic corrections): using the equilibrium constraint $V'(x) = 4\int \frac{\mathrm{d}u}{x-u}\mathrm{d}\rho_V(u)$, one can see that the main term in the Dirichlet form is

$$\frac{1}{N}\int \sum_i \left(\sum_k \frac{1}{x_i - y_k} - \frac{1}{x_i - \gamma_k}\right)^2 \mathrm{d}\mu_{\mathbf{y}}$$
$$\approx \frac{1}{N}\sum_{i=1}^{K}\left(\sum_k \frac{|y_k-\gamma_k|}{|x_i-\gamma_k|^2}\right)^2 \approx \frac{1}{N}\sum_{i=1}^{K}\left(\sum_{k\geq K}\frac{1/N}{((k-i)/N)^2}\right)^2 \approx N\log K.$$

However, the above heuristics do not take into account the extra convexity coming from the pairwise interactions between the x_k's. As a consequence, an improvement is possible if one considers functions of *differences* between eigenvalues. This was first observed by Erdős, Schlein and Yau[14] in the context of Wigner matrices. For our conditional measure, the analogue result is that, for arbitrary small $\varepsilon > 0$, for some c depending on ε,

$$\left(\mathbb{E}_{\mu_{\mathbf{y}}} - \mathbb{E}_{\tilde{\mu}_{\tilde{\mathbf{y}}}}\right)\frac{1}{K}\left(\sum f(N(x_i - x_{i+1}))\right) \leq c\sqrt{\frac{N^\epsilon}{K}\tau \mathrm{D}_{\mu_{\mathbf{y}}}(\sqrt{q})} + c\,e^{-N^\epsilon}\sqrt{\mathrm{S}_{\mu_{\mathbf{y}}}(q)}. \quad (6)$$

The entropy term is negligible due to its exponentially small coefficient, and the extra K^{-1} coefficient in front of the Dirichlet form almost yields the expected convergence to 0. This is however not sufficient, a last ingredient from Ref. 14 being used to speed up the dynsmics of the Dyson Brownian motion, and therefore improve the inequality (6).

Note that there are still major obstructions to making the above heuristics rigorous. One of them is that, for example y_L may be very close to x_{L+1}, or that many y_j's could pile up close to x_L. The contribution to the entropy term, singular of type $1/(x_{L+1} - y_L)^2$, could then be exceptionally large. This problem is taken care of by showing that the local statistics are insensitive to the accumulation of a small number of y_j's. For this, the total variation-entropy inequality can be used. Another obstruction is that the above heuristics strongly depend on the concentration properties of the y_k's around the γ_k's, till the optimal scale N^{-1}, i.e. on showing that the good sets are the generic sets. This is explained in the next section.

4. The rigidity

4.1. *Statement of the result*

The following result states that the ordered particles of one-dimensional log-gases exhibit a strong rigidity.

Theorem 4.1 (B., Erdős, Yau). *Under the assumptions for (2), the bulk particles are concentrated up to scale $N^{-1+\varepsilon}$: for any $\varepsilon > 0$ and $\alpha > 0$, there is a constant $\vartheta > 0$ such that for any N and $k \in [\![\alpha N, (1-\alpha)N]\!]$, we have*

$$\mathbb{P}\left(|\lambda_k - \gamma_k| > N^{-1+\varepsilon}\right) \leq \vartheta^{-1} e^{-N^{\vartheta}}.$$

As we already mentioned, by the Bakry Émery criterion,[?] if $\nabla H^2 > \tau^{-1} > 0$ then$\mu(\mathrm{d}x) = \frac{1}{Z} e^{-H(x)} \mathrm{d}x$ satisfies a logarithmic Sobolev inequality with constant τ. In our case, writing the measure (1) as $e^{-\beta H}$, one finds, for any $\mathbf{v} \in \mathbb{R}^N$,

$$\langle \mathbf{v}, \nabla^2 H \mathbf{v} \rangle = \frac{N}{4} \sum V''(\lambda_i) v_i^2 + \sum_{i<j} \frac{(v_i - v_j)^2}{(\lambda_i - \lambda_j)^2}. \tag{7}$$

As a consequence, if the external potential V is convex, the logarithmic Sobolev inequality with constant of order N yields fluctuations of the λ_i's or order at most $N^{-1/2}$, which is not the optimal $N^{-1+\varepsilon}$. However, for global statistics such as $\sum_{i=1}^N \lambda_i$, the Bakry Émery criterion gives fluctuations of order at most at most 1, which is optimal. Our proof will therefore mainly rely on a proper understanding of convexity at a local scale, as explained in Subsection 4.2.

For this, we first need to introduce the following three definitions (where $\varepsilon > 0$ depends on a).

Rigidity at scale a:	$\mathbb{P}\left(\lvert\lambda_k - \gamma_k\rvert \geq N^{-1+a}\right) \leq \exp(-N^{\epsilon})$,
Concentration at scale a:	$\mathbb{P}\left(\lvert\lambda_k - \mathbb{E}(\lambda_k)\rvert \geq N^{-1+a}\right) \leq \exp(-N^{\epsilon})$,
Accuracy at scale a:	$\lvert\gamma_k^{(N)} - \gamma_k\rvert \leq N^{-1+a}$,

where we remind that γ_k is defined by $\int_{-\infty}^{\gamma_k} \rho_V(s)\mathrm{d}s = \frac{k}{n}$, and $\gamma_k^{(N)}$ is defined by $\int_{-\infty}^{\gamma_k^{(N)}} \rho_1^{(N)}(s)\mathrm{d}s = \frac{k}{n}$. A remarkable fact is that rigidity at scale a for particles in the bulk implies concentration and then accuracy at a better scale:

$$\mathrm{Rig}(a) \xrightarrow{(1)} \mathrm{Conc}(a/2) \xrightarrow{(2)} \mathrm{Acc}(3a/4) \xrightarrow{(3)} \mathrm{Rig}(3a/4).$$

This self-improving scheme allows to conclude the proof of the optimal rigidity by bootstrapping. The step (3) is easy from the definitions. The step (1) relies on a local logarithmic Soblev inequality, explained in the next subsection, and the step (2) relies on the loop equation, explained in Subsection 4.3.

4.2. *The local logarithmic Sobolev inequality.*

The main idea to get an improvement of concentration is that, although the bound $\nabla^2 H \geq N$ cannot be improved, for most directions v the equation (7) gives a much

better lower bound, in particular if $\sum v_i = 0$. More precisely, assume that the positions of the λ_k's are close to γ_k up to an error of order M/N ($M = N^a$ in our induction), and take a I a subset of M successive particles (called $\mathbf{x}$) in the bulk. Then, under the constraint $\sum_I v_i = 0$ equation (7) gives an estimate of order

$$\langle \mathbf{v}, \nabla^2 H \mathbf{v} \rangle \geq \left(\frac{N}{M}\right)^2 \sum_I (v_i - v_j)^2 \geq \frac{N^2}{M} \sum_I v_i^2.$$

This improved convexity bound in this specific directions implies a better concentration estimate thanks to the following lemma, where $\mathbf{x}$ needs to be thought as the internal particles (M of them) and $\mathbf{y}$ as the external ones. This corresponds to Lemma 3.9 in Re. 5.

Lemma 4.1. *Assume $\mu = e^{-(H_1+H_2)}$, where H_1 depends only on $\mathbf{x}$ (M particles), H_2 on $\mathbf{x}$ and $\mathbf{y}$, H_2 convex, and H_1 independent of $\sum x_i$. Suppose that for any $\mathbf{v} \in \mathbb{R}^M$, $\langle v, \nabla^2 H_1 \mathbf{v} \rangle \geq \left(\frac{N}{M}\right)^2 \sum (v_i - v_j)^2$, then for any function of type $f(\mathbf{x}) = F(\sum_1^M v_i x_i)$ (where $\sum v_i = 0$), the following local logarithmic Sobolev inequality holds:*

$$\mathrm{S}_\mu(f) \leq \frac{M}{N^2} \mathrm{D}_\mu(\sqrt{f}).$$

The main tool to prove this concentration result is the Brascamp-Lieb inequality[7] , and it implies that linear statistics of type $\sum_{i:|i-k|<M} v_i \lambda_i$ ($\sum v_i = 0$) have fluctuations at most $\frac{\sqrt{M}}{N}$. As an application, denote $I_k^{(M)} = \{j : |j-k| \leq M\}$, $N^a = M_1 < \cdots < M_\ell = \kappa N$, and $\lambda_k^{(M)} = \frac{\sum_{I_k^{(M)}} \lambda_i}{|I_k^{(M)}|}$. Write

$$\lambda_k = (\lambda_k - \lambda_k^{(M_1)}) + \cdots + (\lambda_k^{(M_{\ell-1})} - \lambda_k^{(M_\ell)}) + \lambda_k^{(M_\ell)}$$

The first term, of type $\sum v_i \lambda_i, \sum v_i = 0$, is concentrated at scale $\sqrt{M_1}/N = N^{-1+\frac{a}{2}}$ by the above Lemma. For a proper choice of the M_i's, the other terms of this telescopic sum have small fluctuations too, and the last one, $\lambda_k^{(\kappa N)}$, has fluctuations at most N^{-1}, as previously observed by a direct application of the Bakry Émery criterion.

This completes the (sketch of) proof for the improvement of concentration of λ_k, by a multiscale analysis. This does not imply the improvement of rigidity, as the distance between $\mathbb{E}(\lambda_k)$ and γ_k may still be of order N^{-1+a}. This is not the case as explained in the next subsection.

4.3. *The loop equation.*

A remarkable fact about β-ensembles is that their Stieltjes transform satisfy a family of equations, called loop-equations, which turn out to be useful in many situations. In our case, we will need the first order loop equation, as introduced by Johansson[18] in a random matrix theory context (see also Eynard[15] , Shcherbina[25]).

We define $m(z) = \int \frac{\rho_V(s)}{z-s}\mathrm{d}s$ and $m_N(z) = \int \frac{\rho_1^{(N)}(s)}{z-s}\mathrm{d}s$, where $z = E + \mathrm{i}\eta$. A good control on $m_N - m$ gives information for the typical locations between the particles: if $|m_N - m| = \mathrm{O}(1)$ for $\eta \geq N^{-1+b}$, $\gamma_k^{(N)} - \gamma_k$, is of order at most N^{-1+b}.

The difference $m_N - m$ satisfies the following quadratic equation, as can be proved by integration by parts for example:

$$(m_N - m)^2(z) + s(z)(m_N - m)(z) = c_N(z)$$

where s is a function depending on ρ_V and

$$c_N(z) \sim \frac{1}{N^2}\mathrm{Var}\left(\sum \frac{1}{z-\lambda_i}\right).$$

The nice fact about this equation is that a bound on the variance (which only relies on the concentration properties of the λ_k's) implies a bound on $m_N - m$ (and therefore of the typical location). More precisely, our improved concentration at scale $N^{-1+\frac{a}{2}}$ gives $c_N = \mathrm{O}(1)$ for $\eta \geq N^{-1+\frac{3}{4}a}$, concluding the sketch of the proof for the improvement of the rigidity.

4.4. *Convexification*

In this section, till now we critically used the convexity of V, in particular in Lemma 4.1, where in the proof the Brascamp Lieb inequality requires $\nabla^2 H_2 > 0$. Indeed, remember that for the β-ensemble, writing $\mu \sim e^{-\beta H}$, with $H = -\sum_{i<j}\log(\lambda_j - \lambda_i) + \frac{N}{4}\sum_i V(\lambda_i)$, we have

$$\langle \mathbf{v}, \nabla^2 H \mathbf{v}\rangle = \frac{N}{4}\sum V''(\lambda_i)v_i^2 + \sum_{i<j}\frac{(v_i - v_j)^2}{(\lambda_i - \lambda_j)^2},$$

Therefore the Hamiltonian is convex only when V is. One can actually circumvent this problem by the following argument from Ref. 6. It involves many technical difficulties, but the main idea is summarized hereafter.

Take

$$\tilde{H} = H + \sum_{k=1}^{\ell} X_\alpha^2,\ X_\alpha = \sum_i g_\alpha^{(N)}(i)(\lambda_i - \gamma_i).$$

For example, if we chose $g_k^{(N)}(i) \sim 1$, this would add convexity along the direction $(1, \ldots, 1)$, direction along which the Hamiltonian H has poor convexity. If we take a sufficient large number of such linear statistics (removing the slow modes along which non-convexity holds, cf. Lemma 3.3 in Ref. 6 for the definition of the $g_\alpha^{(N)}$'s), one can get the following result:

- For ℓ large enough (but independent of N) and a good choice of the $g_\alpha^{(N)}$'s, $\tilde{H}$ is convex, hence optimal bulk rigidity holds for $\tilde{\mu} \sim e^{-N\tilde{H}}$, by the arguments of the previous subsection.

- But under μ, by the loop equation, X_α is of order 1 (these macroscopic statistics were studied in Refs. 18,25), hence rigidity holds for μ as well: for the measures μ and $\tilde{\mu}$ the event with exponentially small probability are the same (cf. Lemma 3.6 in Ref. 6).

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