Homework 1

Due Monday, April 4

Send the homework via email to cfgranda@cims.nyu.edu, do **not** give in a hard copy

1. *Optimization algorithms* Use any programming language of your choice (Matlab, python, etc.) to implement the following methods for least-squares regression with ℓ_1 norm regularization,

$$\text{minimize} \quad \frac{1}{2} \left|\left|Ax - y\right|\right|_2^2 + \lambda \left|\left|x\right|\right|_1, \tag{1}$$

where $A \in \mathbb{R}^{2000 \times 1000}$, $y = Ax' + z$ and x' is 100-sparse. The noise z should be iid Gaussian noise with mean zero and standard deviation equal to 0.1. Sample the entries of A and x' independently at random from a Gaussian distribution with zero mean and standard deviation equal to 1. Set λ to a value that yields a sparse solution.

a. Subgradient descent with a step size equal to α/k for some constant α.

b. ISTA with a constant step size.

c. FISTA with a constant step size.

d. Coordinate descent.

Submit your code and a plot of the convergence of the different methods over the first 100 iterations. Comment briefly on your results.

2. *Total variation* In this problem you will use CVX to study the performance of total-variation regularization for denoising and compressed sensing. CVX allows to solve a large class of convex-optimization problems of small size without worrying about implementation issues.

We will study a toy problem that mimics some of the challenges that arise in image reconstruction from magnetic-resonance imaging (MRI) data. Since images are often reasonably well modeled as being approximate piecewise constant, we consider a 1D piecewise-constant signal. Similarly, MRI measurements can be modeled as samples from the 2D spectrum of a slice of our body in many cases, so we will simulate the data by computing the DFT of the signal. When answering the questions don't forget to submit your code and the figures generated by the script **hw1_pb2.m**.

a. Complete the code in the function **tv_denoising.m** which performs denoising by solving the problem

$$\text{minimize} \quad \frac{1}{2} \left|\left|Fx - y\right|\right|_2^2 + \lambda \ \text{TV}\left(x\right). \tag{2}$$

where x represents the signal, y the data and F the measurement operator. Select three values of the regularization parameter $\lambda > 0$ in the script **hw1_pb2.m** that illustrate what happens when λ is too small, too large or about right. Comment briefly on the results.

b. An important issue in MRI is that sampling the spectrum of the signal is time consuming. This is a problem because the measurement process is costly and because people may have difficulties not moving during the scans (especially children). A possible way of reducing measurement time is to take less measurements. In our simple model, this is equivalent to selecting a subset of the rows of the DFT matrix. As a result, we cannot just invert the matrix to estimate the signal. A possibility is to compute the minimum-ℓ_2-norm estimate by solving

$$\text{minimize} \quad ||x||_2 \quad \text{subject to } Fx = y, \tag{3}$$

where x represents the signal, y the data and F the measurement operator. Complete the code in the function **min_norm_estimate.m** which solves this problem. What result do you observe when you apply it on regularly undersampled data? (The script will plot the estimate for you.)

c. Explain the result you observe in (b) mathematically. Recall that the ℓ_2-norm-minimization problem has a closed-form solution. You will probably find it helpful to write the regularly undersampled DFT matrix F_U as $[F\ F]$ where F is a smaller DFT matrix and the signal as $x = [x_l\ x_r]$, where x_l contains the first half of the entries and x_r the second.

d. Complete the code in the function **min_norm_estimate.m** which solves the problem

$$\text{minimize} \quad \text{TV}(x) \quad \text{subject to } Fx = y. \tag{4}$$

What result do you observe? Does this make sense? Is it reasonable to expect that minimizing the total variation could recover the original signal?

e. Apply both methods to estimate the signal from random measurements (the script already contains the code). Comment on your results briefly comparing them to the results obtained for regular undersampling.