Super-resolution

Optimization-Based Data Analysis

http://www.cims.nyu.edu/~cfgranda/pages/OBDA_spring16

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Super-resolution

- **Optics**: Data-acquisition techniques to overcome the diffraction limit
- **Image processing**: Methods to upsample images onto a finer grid while preserving edges and hallucinating textures
- **This lecture**: Signal estimation from low-resolution data

Super-resolution of point sources
- Spatial super-resolution
- Spectral super-resolution
- Deconvolution in reflection seismography

Conditioning of super-resolution

Linear methods
- Periodogram
- Local fitting

Parametric methods
- Prony's method
- Subspace methods
- Matrix-pencil methods

Super-resolution via convex programming
- Exact recovery
- Super-resolution from noisy data

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  - Spatial super-resolution
  - Spectral super-resolution
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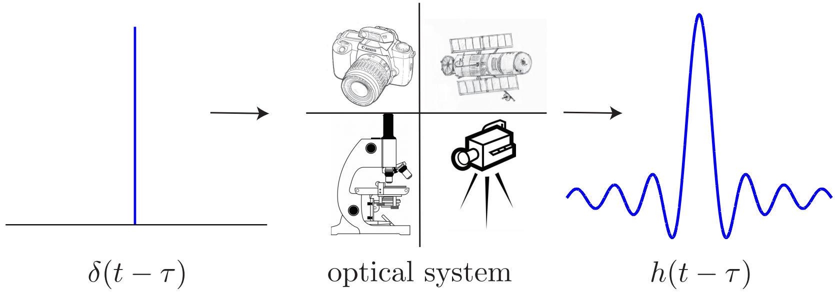
  - Exact recovery
  - Super-resolution from noisy data

Limits of resolution in imaging

The resolving power of lenses, however perfect, is limited (**Lord Rayleigh**)



Diffraction imposes a fundamental limit on the resolution of optical systems

Measurement model

Sensing mechanism acts as a low-pass filter

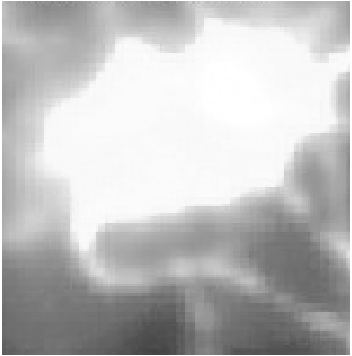
$$x_{\text{LR}} := \phi * x$$

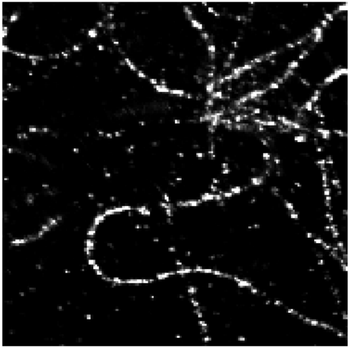
$$\begin{aligned}\widehat{x}_{\text{LR}} &= \widehat{\phi}\,\widehat{x} \\ &= \widehat{\phi}\,\Pi_{[-f_c,f_c]^d}\,\widehat{x}\end{aligned}$$

High-frequency information is gone

We need prior assumptions to recover the signal

Super-resolution of point sources





Mathematical model

- Signal: Superposition of Dirac measures with support T

$$x := \sum_j c_j \delta_{t_j} \qquad c_j \in \mathbb{C},\ t_j \in T \subset [0,1]$$

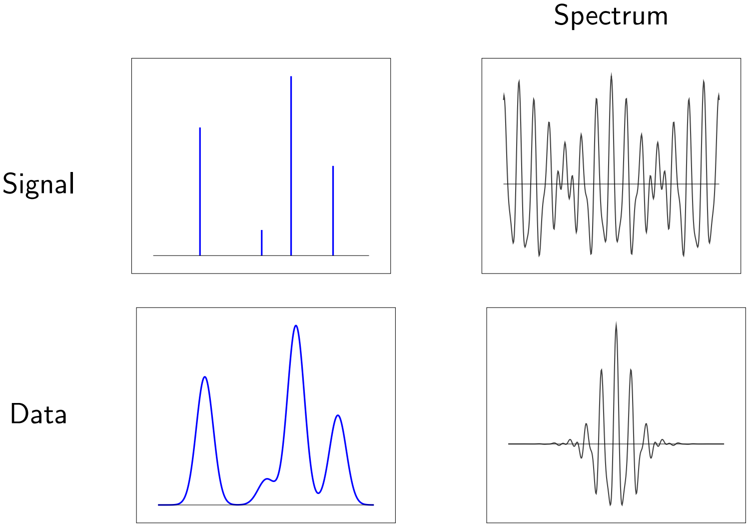
- Data: Convolution of signal and point-spread function

$$\begin{aligned} x_{\mathrm{LR}}(t) &:= \phi * x(t) \\ &= \sum_{t_j \in T} c_j \phi(t - t_j), \end{aligned}$$

Equivalently, low-pass Fourier coeffs with cut-off frequency f_c

$$\begin{aligned} y &= \mathcal{F}_c\, x \\ y_k &= \int_0^1 e^{-i2\pi k t} x(\mathrm{d}t) = \sum_j c_j e^{-i2\pi k t_j}, \quad k \in \mathbb{Z},\ |k| \le f_c \end{aligned}$$

Spatial Super-resolution



Mathematical model

- Signal: Multisinusoidal signal

$$g\left(t\right) := \sum_{f_j \in T} c_j e^{-i2\pi f_j t}$$

$$\widehat{g} = \sum_{f_j \in T} c_j \delta_{f_j}$$

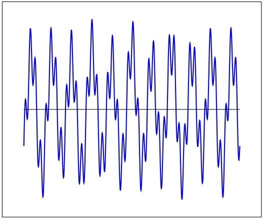
- Data: n samples measured at Nyquist rate

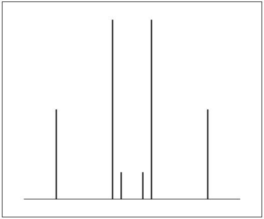
$$g\left(k\right) := \sum_{f_j \in T} c_j e^{-i2\pi k f_j}, \qquad 1 \leq k \leq n$$

Spectral Super-resolution

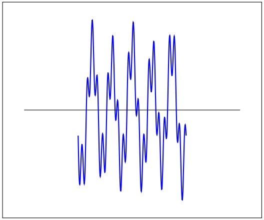
Spectrum

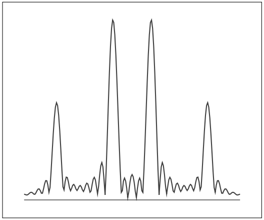
Signal





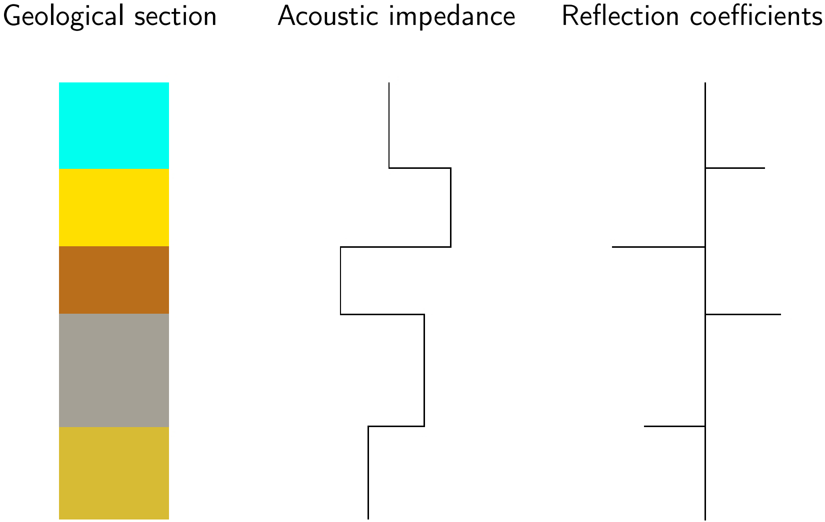
Data



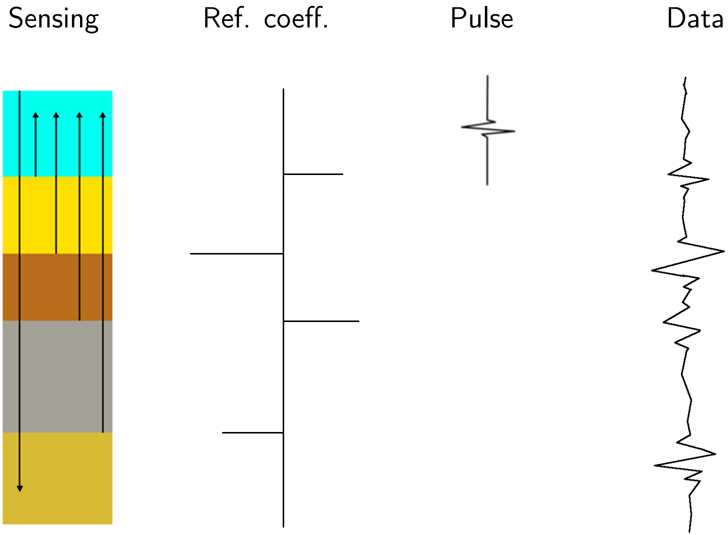


Seismology

Reflection seismology

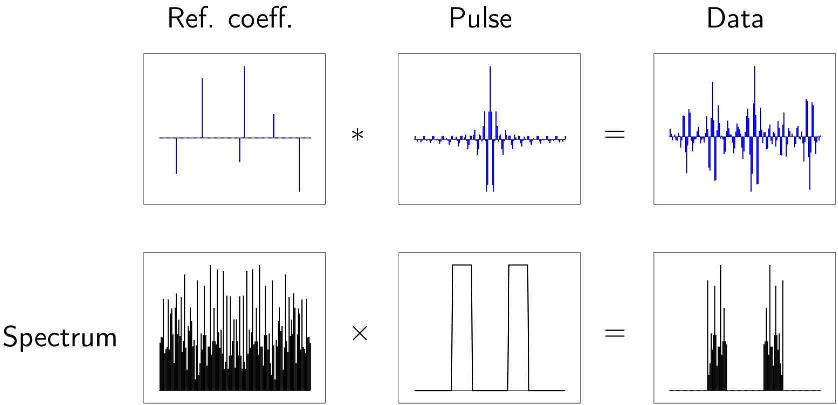


Reflection seismology



Data ≈ convolution of pulse and reflection coefficients

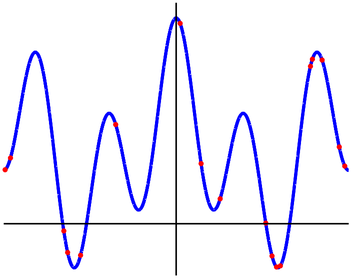
Sensing model for reflection seismology



Compressed sensing vs super-resolution

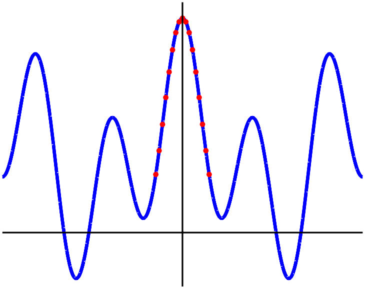
Estimation of sparse signals from undersampled measurements suggests connections to compressed sensing

Compressed sensing



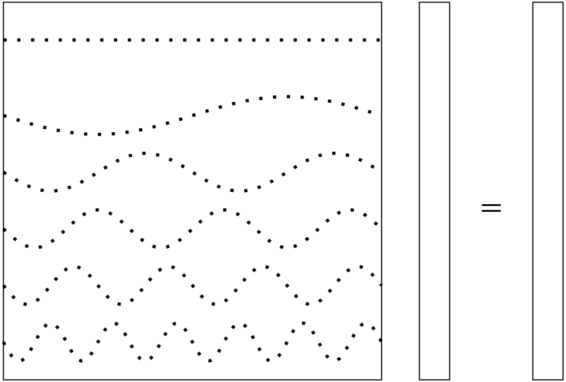
spectrum interpolation

Super-resolution



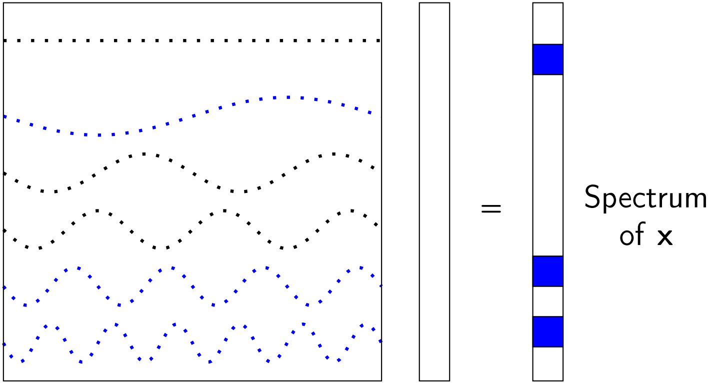
spectrum extrapolation

Compressed sensing



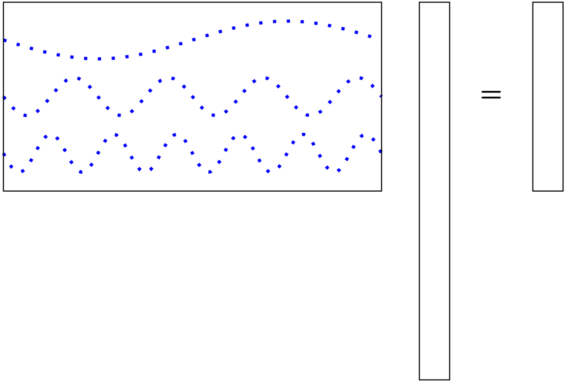
Spectrum of $\mathbf{x}$

Compressed sensing

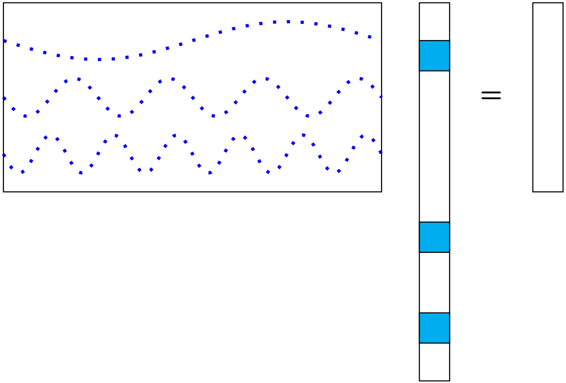


Measurement operator = random frequency samples

Compressed sensing

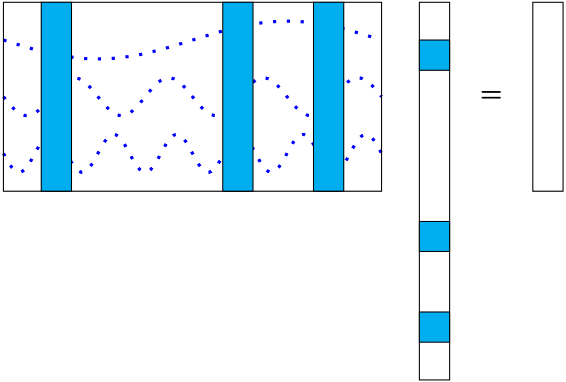


Compressed sensing



Aim: Study effect of measurement operator on sparse vectors

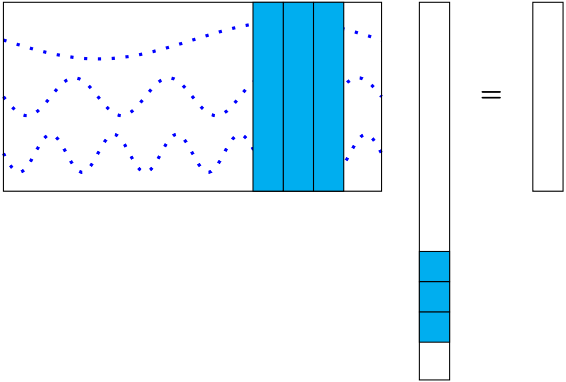
Compressed sensing



Operator is well conditioned when acting upon any sparse signal

(*restricted isometry property*)

Compressed sensing



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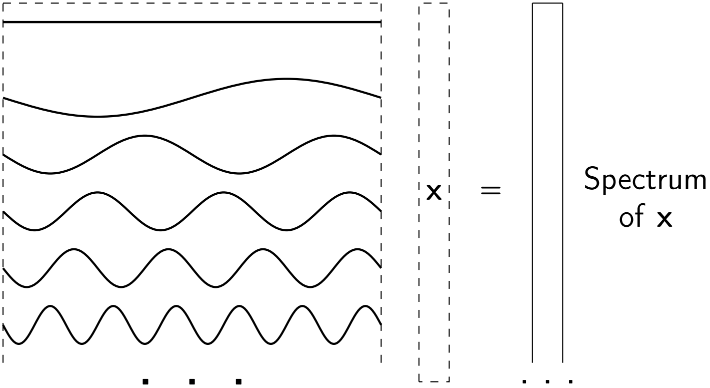
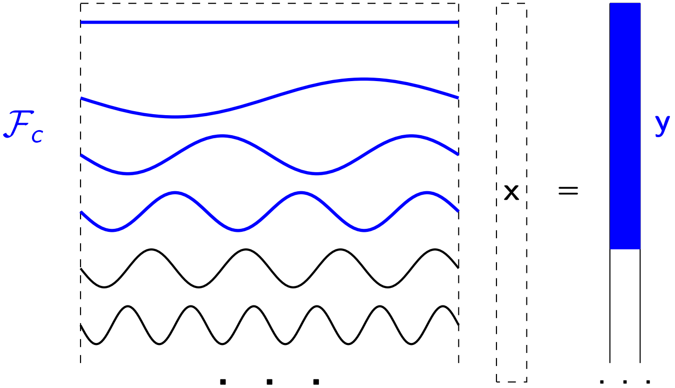
Super-resolution



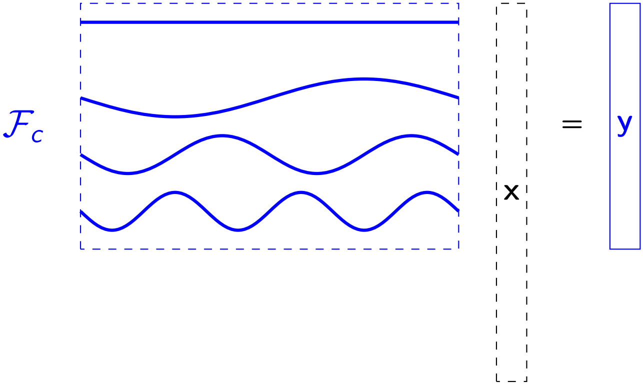
No discretization

Super-resolution



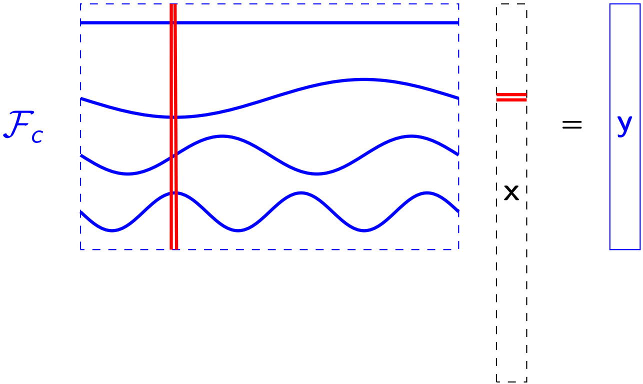
Data: Low-pass Fourier coefficients

Super-resolution



Data: Low-pass Fourier coefficients

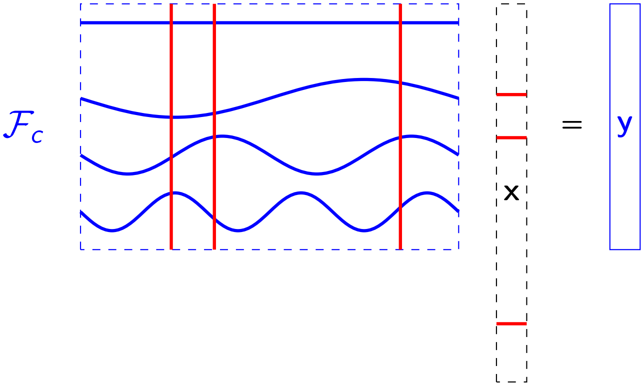
Super-resolution



Problem: If the support is clustered, the problem may be ill posed

In super-resolution sparsity is not enough!

Super-resolution



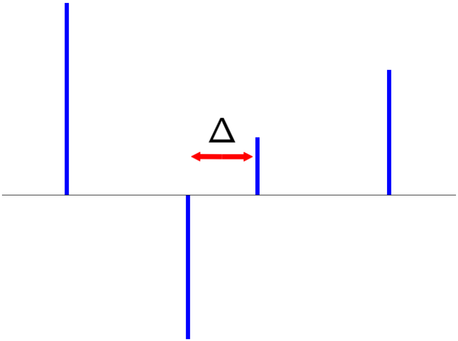
If the support is spread out, there is still hope

We need conditions beyond sparsity

Minimum separation

The minimum separation Δ of a discrete set T is

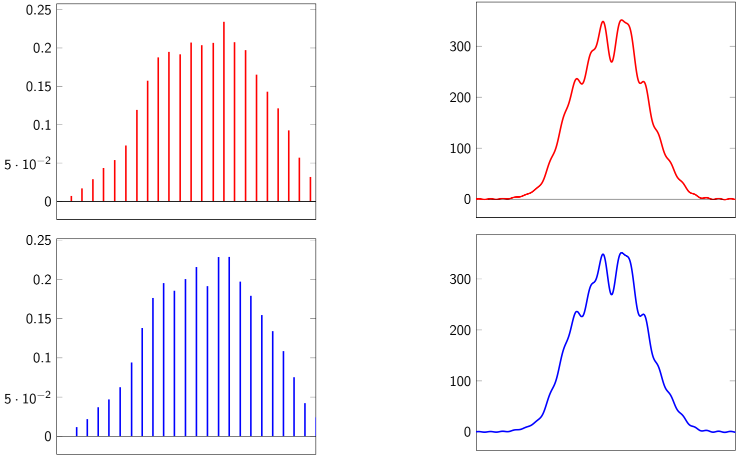
$$\Delta = \inf_{(t,t') \in T \,:\, t \neq t'} |t - t'|$$



Example: 25 spikes, $f_c = 10^3$, $\Delta = 0.8/f_c$

Signals

Data (in signal space)



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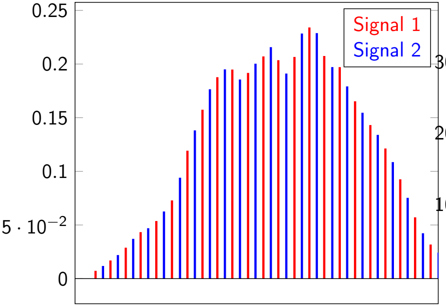
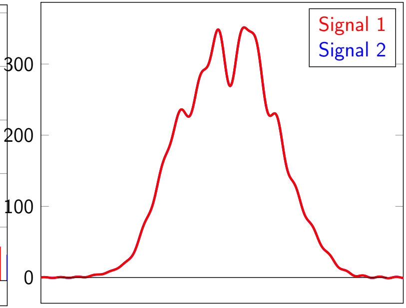


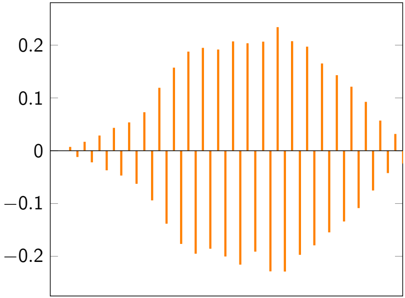
Signals



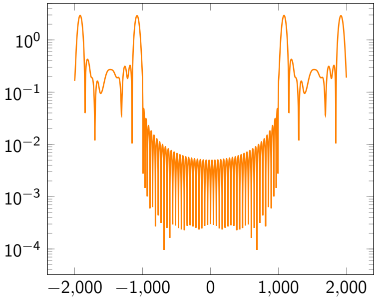
Data (in signal space)

Example: 25 spikes, $f_c = 10^3$, $\Delta = 0.8/f_c$

The difference is almost in the null space of the measurement operator



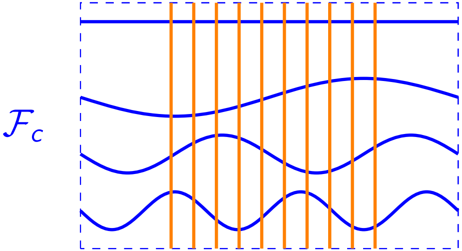
Difference



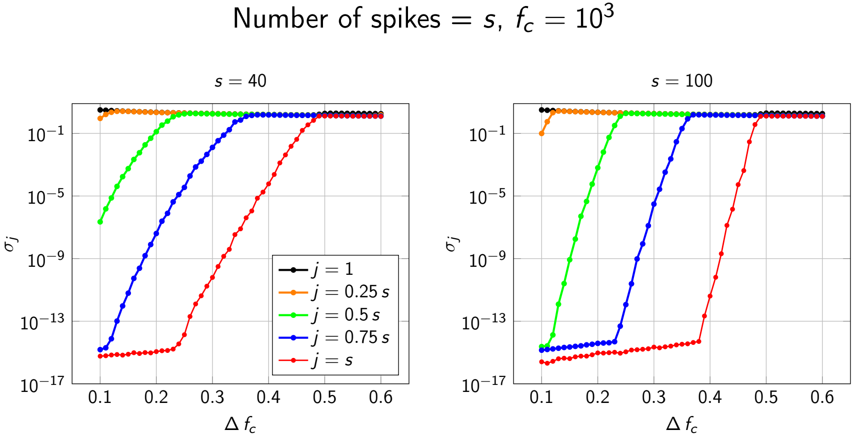
Spectrum

Lower bound on Δ

- Above what minimum distance Δ is the problem well posed?
- Numerical lower bound on Δ:
  1. Compute singular values of restricted operator for different values of Δ_{diff}
  2. Find Δ_{diff} under which the restricted operator is ill conditioned
  3. Then $\Delta \geq 2\Delta_{\text{diff}}$



Singular values of the restricted operator

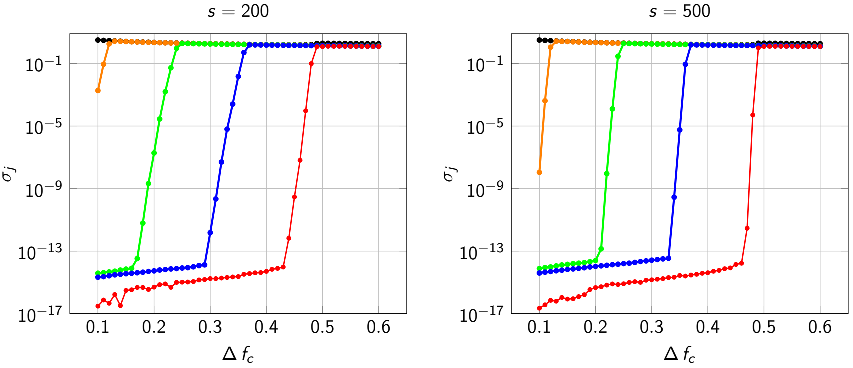


Phase transition at $\Delta_{\text{diff}} = 1/2f_c \rightarrow \Delta = 1/f_c$

Characterized asymptotically by Slepian's *prolate spheroidal sequences*

Singular values of the restricted operator

Number of spikes $= s$, $f_c = 10^3$

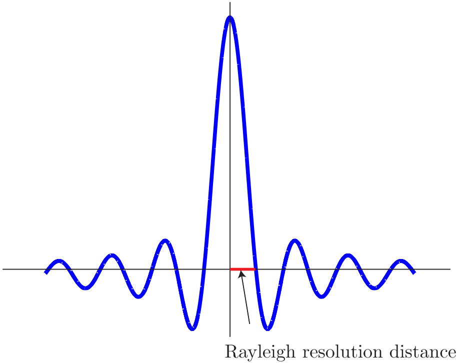


Phase transition at $\Delta_{\text{diff}} = 1/2f_c \rightarrow \Delta = 1/f_c$

Characterized asymptotically by Slepian's *prolate spheroidal sequences*

Interpretation of $\lambda_c := 1/f_c$

Diameter of point-spread function



$\lambda_c/2$ is the Rayleigh resolution distance

Periodogram

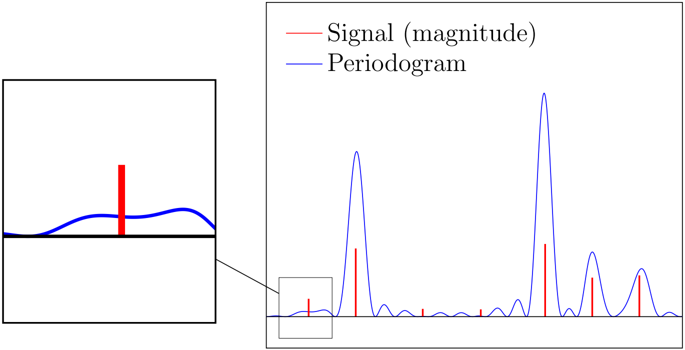
Spectrum of truncated data in spectral super-resolution

$$P(t) = \mathcal{F}_n^* y$$
$$= \sum_{t_j \in T} c_j D_{f_c}(t - t_j),$$

D_{f_c} is the periodized sinc or Dirichlet kernel

$$D_{f_c}(t) := \sum_{k=-f_c}^{f_c} e^{i2\pi kt} = \begin{cases} 1 & \text{if } t = 0 \\ \frac{\sin((2f_c+1)\pi t)}{(2f_c+1)\sin(\pi t)} & \text{otherwise} \end{cases}$$

Periodogram



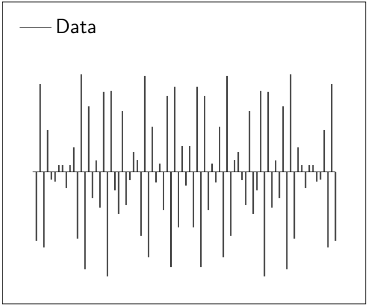
Windowing

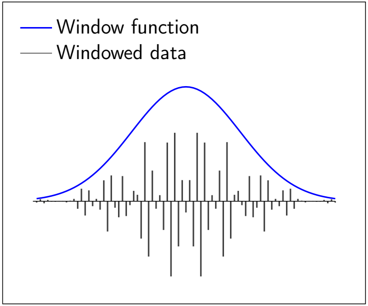
Window function $\widehat{w} \in \mathbb{C}^n$

$$y_{\widehat{w}} = y \cdot \widehat{w}$$

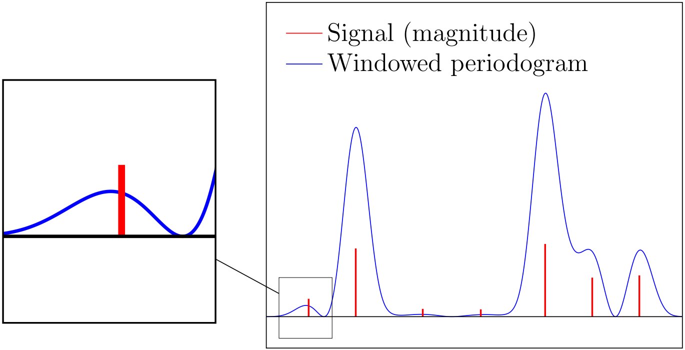
$$\begin{aligned} P_{\widehat{w}}(f) &= \mathcal{F}_n^* \, y_{\widehat{w}} \\ &= \sum_{t_j \in T} c_j w\left(t - t_j\right), \end{aligned}$$

Windowing



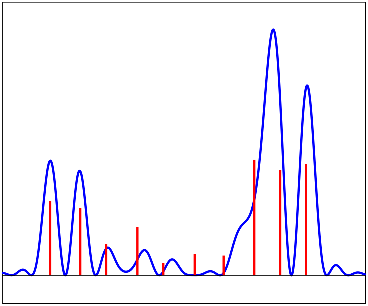


Windowing

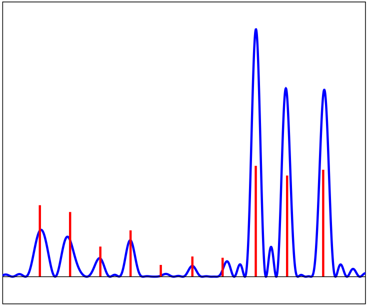


Minimum separation: Periodogram

$\Delta = \frac{0.6}{f_c}$

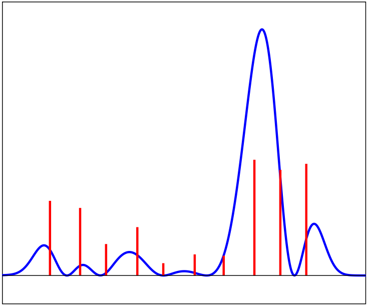


$\Delta = \frac{1.2}{f_c}$

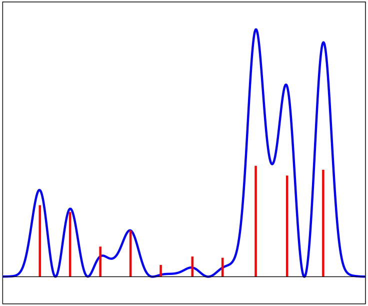


Minimum separation: Gaussian periodogram

$$\Delta = \frac{0.6}{f_c}$$



$$\Delta = \frac{1.2}{f_c}$$



Local fitting

Assume only one source

$$x_{\mathrm{LR}}(t) := c_1 \phi(t - t_1).$$

Estimation via best ℓ_2-norm fit

$$\begin{aligned} t_{\mathrm{est}} &= \arg\min_{\tilde{t}} \min_{\alpha \in \mathbb{C}} \left|\left| x_{\mathrm{LR}} - \alpha\, \phi_{\tilde{t}} \right|\right|_2 \\ &= \arg\max_{\tilde{t}} \left| \langle x_{\mathrm{LR}}, \phi_{\tilde{t}} \rangle \right| \end{aligned}$$

If sources are far we can compute local fits

Equivalent to matching pursuit

Prony polynomial

Signal

$$x := \sum_j c_j \delta_{t_j} \qquad c_j \in \mathbb{C},\ t_j \in T \subset [0,1]$$

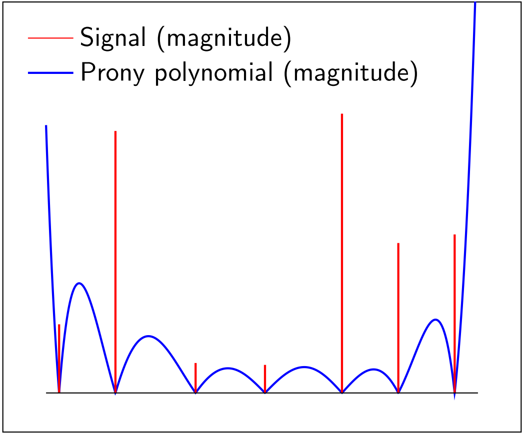
Data

$$y_k := \widehat{x}(k), \qquad 0 \le k \le n-1.$$

Prony polynomial

$$\begin{aligned} P_{\text{prony}}(t) &:= \prod_{j=1}^{s} \left(1 - e^{i2\pi(t-t_j)}\right) \\ &= 1 + \sum_{l=1}^{s} v_l \, e^{i2\pi l t}, \quad v_0 := 1 \end{aligned}$$

Prony polynomial



Computing the Prony polynomial

By construction

$$\langle P_{\text{prony}}, x \rangle = 0$$

By Parseval's Theorem

$$\begin{aligned}\langle P_{\text{prony}}, x \rangle &= \langle v, \widehat{x} \rangle \\ &= \sum_{k=0}^{s} v_k \, \overline{y_k} \qquad \text{if } s+1 \leq n\end{aligned}$$

Computing the Prony polynomial

By construction

$$\left\langle P_{\text{prony}}, e^{2\pi k' t} x \right\rangle = 0$$

By Parseval's Theorem

$$\begin{aligned}\left\langle P_{\text{prony}}, e^{2\pi k' t} x \right\rangle &= \langle v, \widehat{x}_{k'} \rangle \\ &= \sum_{k=0}^{s} v_k \overline{y_{k+k'}} \qquad \text{if } s + k' \leq n - 1\end{aligned}$$

Prony's method

1. Form the system of equations

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_s \\ y_2 & y_3 & \cdots & y_{s+1} \\ \cdots & \cdots & \cdots & \cdots \\ y_s & y_{s+1} & \cdots & y_{n-1} \end{bmatrix} \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \cdots \\ \tilde{v}_s \end{bmatrix} = - \begin{bmatrix} y_0 \\ y_1 \\ \cdots \\ y_{s-1} \end{bmatrix}$$

2. Solve the system and set $\tilde{v}_0 = 1$

3. Roots of polynomial with coeffs $\tilde{v}_0, \ldots, \tilde{v}_s$: $z_1, \ldots, z_s$

4. For $z_j = e^{i2\pi\tau}$ include τ in estimated support

Prony's method

$$\begin{bmatrix} y_1 & y_2 & \dots & y_s \\ y_2 & y_3 & \dots & y_{s+1} \\ \dots & \dots & \dots & \dots \\ y_s & y_{s+1} & \dots & y_{n-1} \end{bmatrix} = \begin{bmatrix} e^{-i2\pi t_1} & e^{-i2\pi t_2} & \dots & e^{-i2\pi t_s} \\ e^{-i2\pi 2t_1} & e^{-i2\pi 2t_2} & \dots & e^{-i2\pi 2t_s} \\ \dots & \dots & \dots & \dots \\ e^{-i2\pi s t_1} & e^{-i2\pi s t_2} & \dots & e^{-i2\pi s t_s} \end{bmatrix}$$

$$\begin{bmatrix} c_1 & 0 & \dots & 0 \\ 0 & c_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & c_s \end{bmatrix}$$

$$\begin{bmatrix} 1 & e^{-i2\pi t_1} & \dots & e^{-i2\pi(s-1)t_1} \\ 1 & e^{-i2\pi t_2} & \dots & e^{-i2\pi(s-1)t_2} \\ \dots & \dots & \dots & \dots \\ 1 & e^{-i2\pi t_s} & \dots & e^{-i2\pi(s-1)t_s} \end{bmatrix}$$

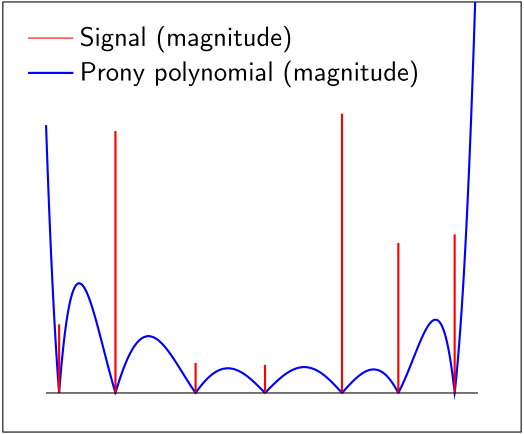
Vandermonde matrix

For any distinct s nonzero $z_1, z_2, \ldots, z_s \in \mathbb{C}$ and any m_1, m_2, s such that $m_2 - m_1 + 1 \geq s$

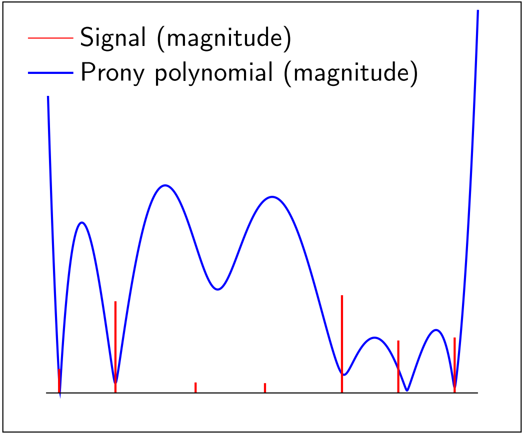
$$\begin{bmatrix} z_1^{m_1} & z_2^{m_1} & \cdots & z_s^{m_1} \\ z_1^{m_1+1} & z_2^{m_1+1} & \cdots & z_s^{m_1+1} \\ z_1^{m_1+2} & z_2^{m_1+2} & \cdots & z_s^{m_1+2} \\ & & \cdots & \\ z_1^{m_2} & z_2^{m_2} & \cdots & z_s^{m_2} \end{bmatrix}$$

is full rank

No noise



SNR = 140 dB (relative ℓ_2 norm of noise = 10^{-8})



Alternative interpretation of Prony's method

Prony's method finds nonzero vector in the null space of $Y(s+1)^T$

$$Y(m) := \begin{bmatrix} y_0 & y_1 & \cdots & y_{n-m} \\ y_1 & y_2 & \cdots & y_{n-m+1} \\ \cdots & \cdots & \cdots & \cdots \\ y_{m-1} & y_m & \cdots & y_{n-1} \end{bmatrix}$$

The vector corresponds to the coefficients of the Prony polynomial

Notation: Sinusoidal atoms

For $k > 0$

$$a_{0:k}(t) := \begin{bmatrix} 1 \\ e^{-i2\pi t} \\ e^{-i2\pi 2t} \\ \cdots \\ e^{-i2\pi kt} \end{bmatrix}$$

$$A_{0:k}(T) := \begin{bmatrix} a_{0:k}(t_1) & a_{0:k}(t_2) & \cdots & a_{0:k}(t_s) \end{bmatrix}$$

Decomposition

$$Y(m) = \begin{bmatrix} a_{0:m-1}(t_1) & a_{0:m-1}(t_2) & \cdots & a_{0:m-1}(t_s) \end{bmatrix}$$

$$\begin{bmatrix} c_1 & 0 & \cdots & 0 \\ 0 & c_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & c_s \end{bmatrix} \begin{bmatrix} a_{0:n-m}(t_1)^T \\ a_{0:n-m}(t_2)^T \\ \cdots \\ a_{0:n-m}(t_s)^T \end{bmatrix}$$

$$= A_{0:m-1}(T)\; C\, A_{0:m}(T)^T$$

Idea: To estimate T find $a_{0:m-1}(t)$ in the column space of $Y(m)$

Pseudospectrum

To find atoms that are *close* to the column space of $Y(m)$

- Compute orthogonal complement $\mathcal{N}$ of column space of $Y(m)$
- Locate local maxima of pseudospectrum

$$P_{\mathcal{N}}(t) = \log \frac{1}{\left|\mathcal{P}_{\mathcal{N}}\left(a_{0:m-1}(t)\right)\right|^2}$$

Empirical covariance matrix

$\mathcal{N}$ is the null space of the empirical covariance matrix

$$\Sigma(m) = \frac{1}{n-m+1} YY^*$$
$$= \frac{1}{n-m+1} \sum_{j=0}^{n-m} \begin{bmatrix} y_j \\ y_{j+1} \\ \cdots \\ y_{j+m-1} \end{bmatrix} \begin{bmatrix} \overline{y_j} & \overline{y_{j+1}} & \cdots & \overline{y_{j+m-1}} \end{bmatrix}$$

Pseudospectrum

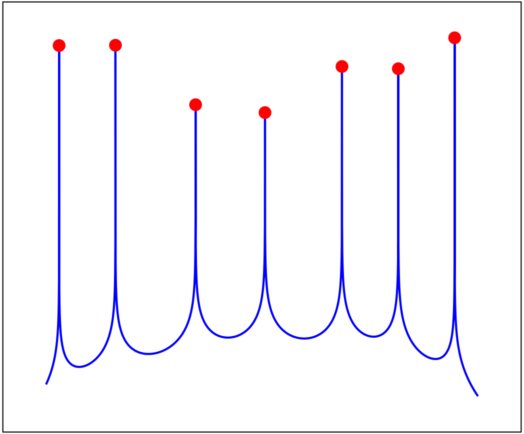
$$Y(m) = A_{0:m-1}(T)\; C\, A_{0:m}(T)^T$$

implies

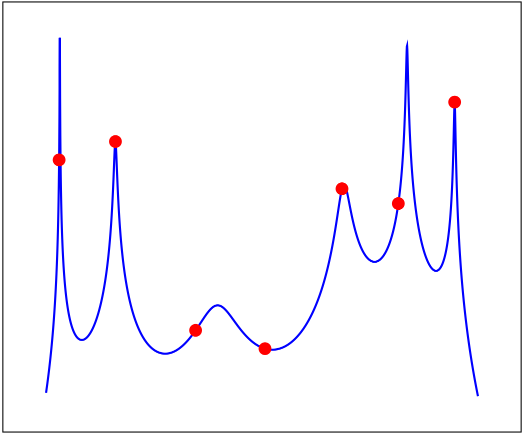
$$P_{\mathcal{N}}(t_j) = \infty, \qquad \text{for } t_j \in T$$

$$P_{\mathcal{N}}(t) < \infty, \qquad \text{for } t \notin T$$

Pseudospectrum: No noise



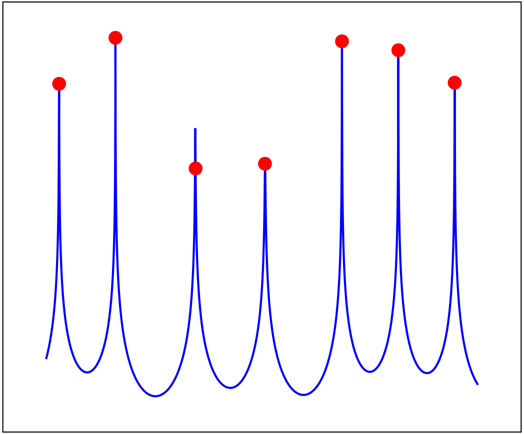
Pseudospectrum: SNR = 140 dB, $n = 2s$



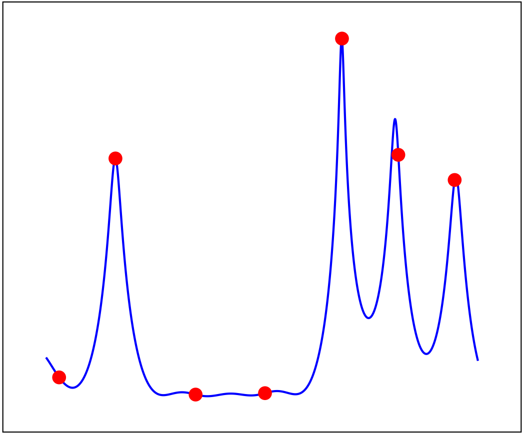
Multiple-signal classification (MUSIC)

1. Build the empirical covariance matrix $\Sigma(m)$
2. Compute the eigendecomposition of $\Sigma(m)$
3. Select $U_{\mathcal{N}}$ corresponding to $m-s$ smallest eigenvalues
4. Estimate support by computing the pseudospectrum

Pseudospectrum: SNR = 40 dB, $n = 81$, $m = 30$



Pseudospectrum: SNR = 1 dB, $n = 81$, $m = 30$



Probabilistic model: Signal

$$x = \sum_{t_j \in T} c_j \delta_{t_j} = \sum_{t_j \in T} |c_j| \, e^{i\phi_j} \delta_{t_j},$$

The phases ϕ_j are independent and uniformly distributed in $[0, 2\pi]$

$$\mathrm{E}(x) = 0$$

$$\mathrm{E}[cc^*] = D_c := \begin{bmatrix} |c_1|^2 & 0 & \dots & 0 \\ 0 & |c_2|^2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & |c_s|^2 \end{bmatrix}$$

Probabilistic model: Noise

Noise z is a zero-mean Gaussian vector with covariance $\sigma^2 I$

$$\begin{aligned}\tilde{y}_k &:= \int_0^1 e^{-i2\pi kt} x(\mathrm{d}t) + z_k \\ &= \sum_{t_j \in T} c_j e^{-i2\pi k t_j} + z_k\end{aligned}$$

$$\tilde{y} = A_{0:m-1}\left(T\right) c + z,$$

Covariance matrix of the data

$$\mathrm{E}\left[\tilde{y}\tilde{y}^*\right] = A_{1:m} D_c A_{1:m}^* + \sigma^2 I$$

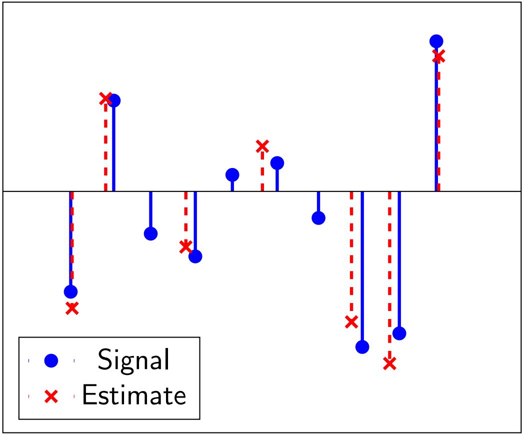
Eigendecomposition of covariance matrix

Eigendecomposition of $\mathrm{E}\left[\tilde{y}\tilde{y}^*\right]$

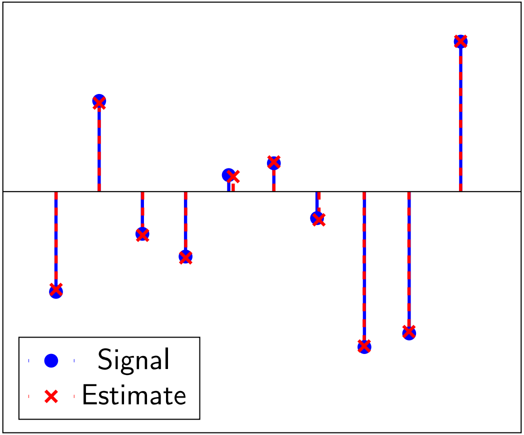
$$\mathrm{E}\left[\tilde{y}\tilde{y}^*\right] = \begin{bmatrix} U_{\mathcal{S}} & U_{\mathcal{N}} \end{bmatrix} \begin{bmatrix} \Lambda + \sigma^2 I_s & 0 \\ 0 & \sigma^2 I_{n-s} \end{bmatrix} \begin{bmatrix} U_{\mathcal{S}}^* \\ U_{\mathcal{N}}^* \end{bmatrix},$$

- $U_{\mathcal{S}} \in \mathbb{C}^{m \times s}$: unitary matrix that spans column space of $A_{1:m}$
- $U_{\mathcal{N}} \in \mathbb{C}^{m \times (m-s)}$: unitary matrix spanning the orthogonal complement
- $\Lambda \in \mathbb{C}^{k \times k}$ is a diagonal matrix with positive entries

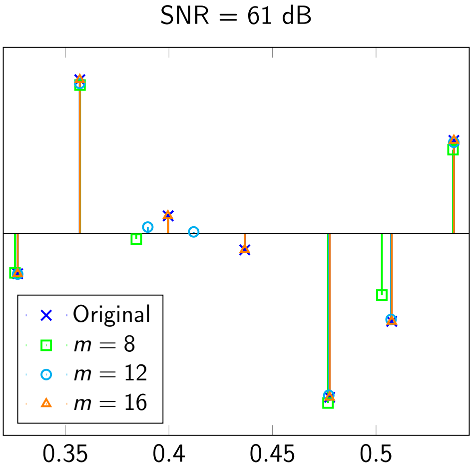
$\Delta = \frac{0.6}{f_c}$, SNR = 20 dB, $n = 81$, $m = 40$



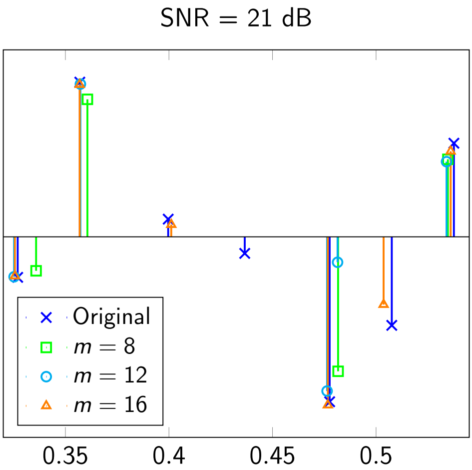
$\Delta = \frac{1.2}{f_c}$, SNR = 20 dB, $n = 81$, $m = 40$



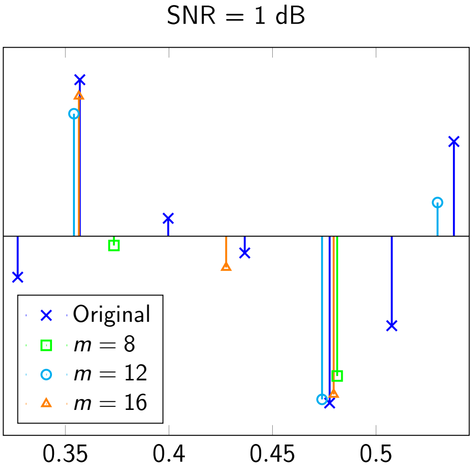
Different values of m



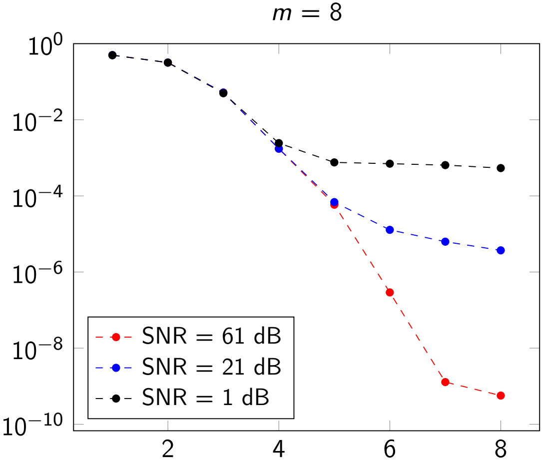
Different values of m



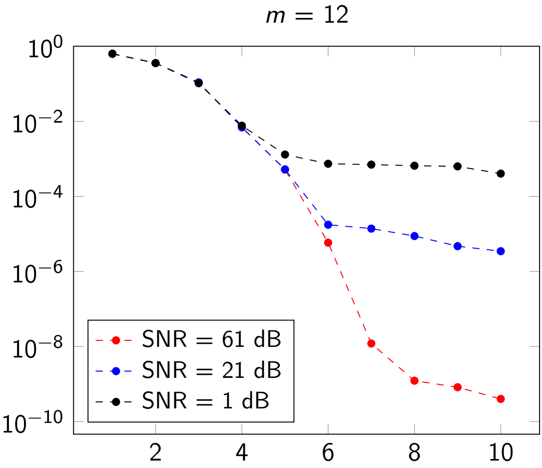
Different values of m



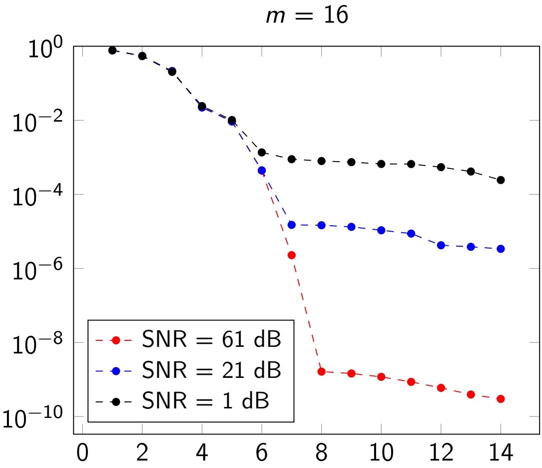
Eigenvalues



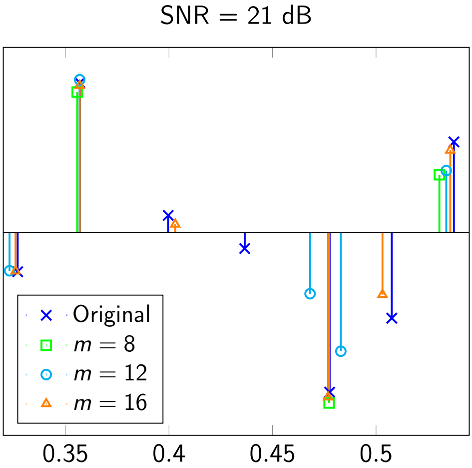
Eigenvalues



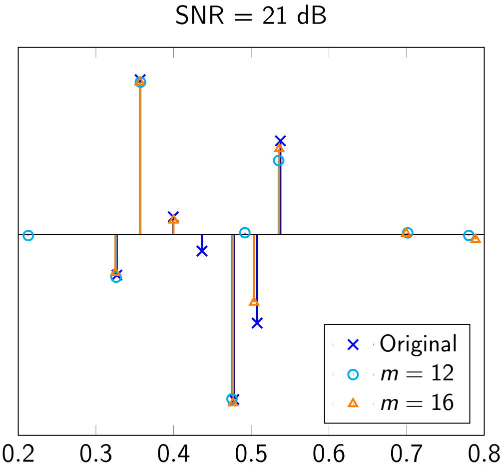
Eigenvalues



Wrong s ($s - 1$)



Wrong s $(s+1)$



Low-rank model

$$Y_0 = \begin{bmatrix} y_0 & y_1 & \cdots & y_{n-m} \\ y_1 & y_2 & \cdots & y_{n-m+1} \\ \cdots & \cdots & \cdots & \cdots \\ y_{m-1} & y_m & \cdots & y_{n-1} \end{bmatrix}$$

$$= A_{0:m-1}(T)\ C\, A_{0:n-m}(T)^T$$

$$= \sum_{t_j \in T} c_j\, a_{0:m-1}(t_j)\ a_{0:n-m}(t_j)^T$$

Matrix pencil

The matrix pencil of two matrices M_1, M_2 is

$$L_{M_1,M_2}(\mu) := M_2 - \mu M_1, \quad \mu \in \mathbb{C}$$

The set of rank-reducing values $\mathcal{R}$ of a matrix pencil satisfy

$$\text{rank}\left(L_{M_1,M_2}(\mu)\right) = \text{rank}\left(L_{M_1,M_2}(\mu_j)\right) + 1$$

for all $\mu_j \in \mathcal{R}$ and $\mu \notin \mathcal{R}$

Matrix pencil

We consider the matrix-pencil of Y_0 and

$$Y_1 = \begin{bmatrix} y_1 & y_2 & \cdots & y_{n-m+1} \\ y_2 & y_3 & \cdots & y_{n-m+2} \\ \cdots & \cdots & \cdots & \cdots \\ y_m & y_{m+1} & \cdots & y_n \end{bmatrix}$$

$$= A_{1:m}(T)\; C\, A_{0:n-m}(T)^T$$

$$= \sum_{t_j \in T} c_j\, a_{1:m}(t_j)\; a_{0:n-m}(t_j)^T$$

$\exp(i2\pi\tau)$ is a rank-reducing value of L_{Y_0,Y_1} if and only if $\tau \in T$

Computing the rank-reducing values

Let $Y_0 = U_0 \, \Sigma_0 \, V_0^*$ be the singular-value decomposition of Y_0

The s eigenvalues of the matrix

$$M = V_0 \, \Sigma_0^{-1} \, U_0^* \, Y_1$$

are equal to $\exp\left(i 2\pi t_j\right)$ for $1 \leq j \leq s$

Proof

$$a_{1:m}(\tau) = \exp(i2\pi\tau)\, a_{0:m-1}(\tau)$$

$$A_{0:m-1}(T) = A_{0:m-1}(T)\,\Phi$$

$$\Phi := \begin{bmatrix} e^{i2\pi t_1} & 0 & \dots & 0 \\ 0 & e^{i2\pi t_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{i2\pi t_s} \end{bmatrix}$$

Proof

$$Y_0 = A_{0:m-1}(T)\, C A_{0:n-m}(T)^T$$

$$C\, A_{0:n-m}(T)^T = U\, \Sigma\, V^*$$

$$C\, A_{0:n-m}(T)^T\, V\, \Sigma^{-1}\, U^* = I$$

$$\begin{aligned}
Y_1 &= A_{1:m}(T)\; C\, A_{0:n-m}(T)^T \\
&= A_{0:m-1}(T)\; \Phi\; C\, A_{0:n-m}(T)^T \\
&= A_{0:m-1}(T)\; C\, A_{0:n-m}(T)^T\, V\, \Sigma^{-1}\, U^* \Phi\; C\, A_{0:n-m}(T)^T \\
&= Y_0\, V\, \Sigma^{-1}\, U^*\, \Phi\, U\, \Sigma\, V^*
\end{aligned}$$

Proof

$$V_0\,\Sigma_0^{-1}\,U_0^*\;Y_0 V = V_0 V_0^* V = V$$

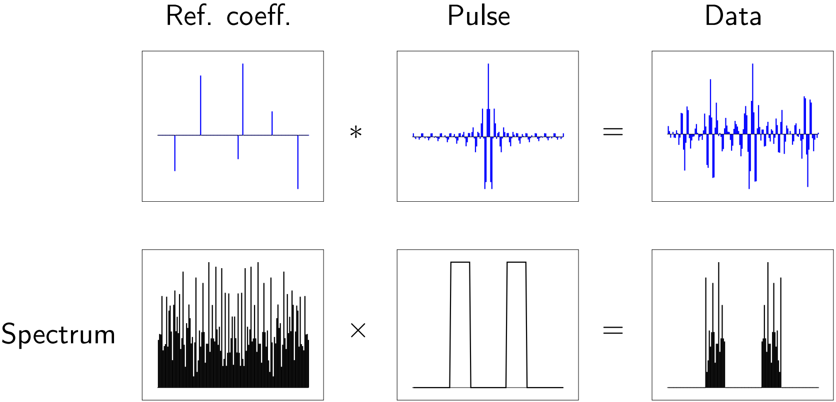
$$\begin{aligned} V_0\,\Sigma_0^{-1}\,U_0^*\;Y_1 &= V_0\,\Sigma_0^{-1}\,U_0^* Y_0\,V\,\Sigma^{-1}\,U^*\,\Phi\,U\,\Sigma\,V^* \\ &= V\,\Sigma^{-1}\,U^*\,\Phi\,U\,\Sigma\,V^* \\ &= P^{-1}\begin{bmatrix}\Phi & 0\\ 0 & 0\end{bmatrix}P \end{aligned}$$

$$P := \begin{bmatrix}U & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}\Sigma & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}V^*\\ V_\perp^*\end{bmatrix}$$

Spectral super-resolution via matrix pencil

1. Build $Y_0 = U_0 \, \Sigma_0 \, V_0^*$ and Y_1
2. Compute the s largest eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$ of $V_0 \, \Sigma_0^{-1} \, U_0^* \, Y_1$
3. Output the phase of $\lambda_1, \lambda_2, \ldots, \lambda_s$ divided by 2π

Sensing model for reflection seismology



Suggestion of various geophysicists: Minimize ℓ_1 norm

Deconvolution with the ℓ_1 norm

Howard L. Taylor,* Stephen C. Banks,‡ and John F. McCoy§

GEOPHYSICS, VOL. 44, NO. 1 (JANUARY 1979)

LINEAR INVERSION OF BAND-LIMITED REFLECTION SEISMOGRAMS*

FADIL SANTOSA† AND WILLIAM W. SYMES‡

SIAM J. SCI. STAT. COMPUT.
Vol. 7, No. 4, October 1986

Reconstruction of a sparse spike train from a portion of its spectrum and application to high-resolution deconvolution

Shlomo Levy* and Peter K. Fullagar‡

GEOPHYSICS, VOL. 46, NO. 9 (SEPTEMBER 1981)

ROBUST MODELING WITH ERRATIC DATA†

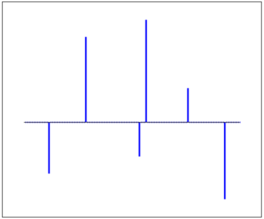
JON F. CLAERBOUT* AND FRANCIS MUIR‡

GEOPHYSICS, VOL. 38, NO. 5 (OCTOBER 1973)

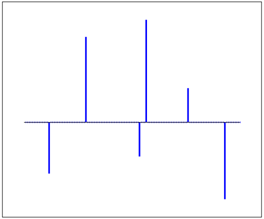
Minimum ℓ_1-norm estimate

$$\begin{array}{ll} \textit{minimize} & ||\text{estimate}||_1 \\ \textit{subject to} & \text{estimate} * \text{pulse} = \text{data} \end{array}$$

Reflection coefficients



Estimate



Total-variation norm

- Continuous counterpart of the ℓ_1 norm
- If $x = \sum_j c_j \delta_{t_j}$ then $||x||_{\mathsf{TV}} = \sum_j |c_j|$
- Not the total variation of a piecewise-constant function
- **Formal definition:** For a complex measure ν

$$||\nu||_{\mathsf{TV}} = \sup_{||f||_\infty \leq 1, f \in C(\mathbb{T})} \int_{\mathbb{T}} \overline{f(t)} x(\mathrm{d}t)$$

Super-resolution via convex programming

For data of the form $y = \mathcal{F}_c\, x$, we solve

$$\min_{\tilde{x}} ||\tilde{x}||_{\text{TV}} \quad \text{subject to} \quad \mathcal{F}_c\, \tilde{x} = y$$

over all finite complex measures $\tilde{x}$ supported on $[0, 1]$

Exact recovery is guaranteed if $\Delta \geq \frac{1.26}{f_c}$

Dual certificate

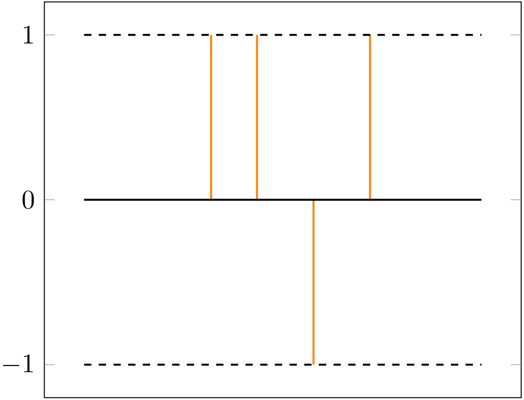
The same as for the ℓ_1 norm, but now q is a function

$$q := \mathcal{F}_c^* v$$

$$\begin{aligned} q_i &= \operatorname{sign}(x_i) && \text{if } x_i \neq 0 \\ |q_i| &< 1 && \text{if } x_i = 0 \end{aligned}$$

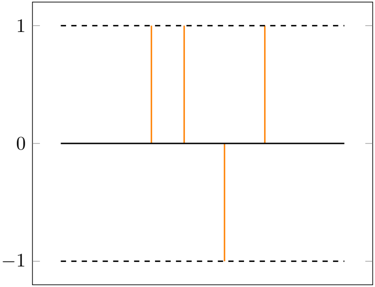
The rows of $\mathcal{F}_c$ are low pass sinusoids instead of random sinusoids

Certificate for super-resolution



Aim: Interpolate sign pattern

Certificate for super-resolution



1st idea: Interpolation with a low-frequency fast-decaying kernel K

$$c(t) = \sum_{i : x_i \neq 0} \alpha_i \, K(t - i)$$

Certificate for super-resolution

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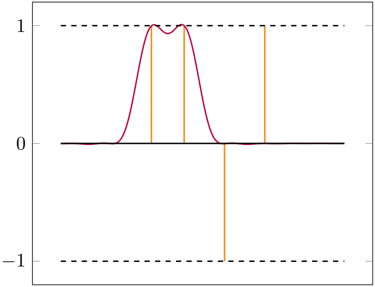

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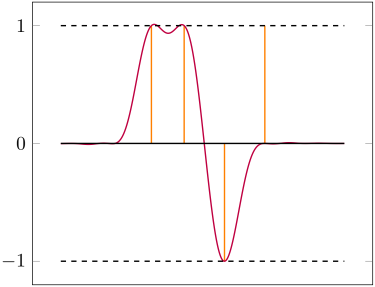

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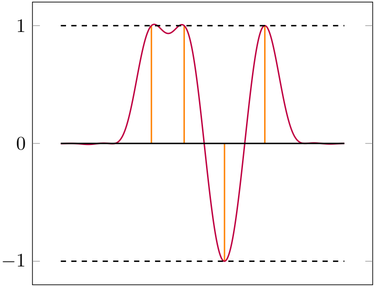

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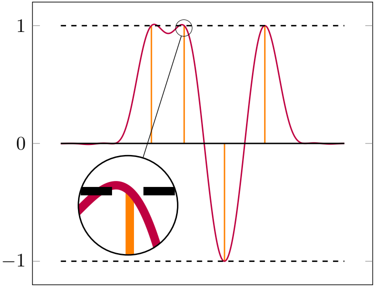
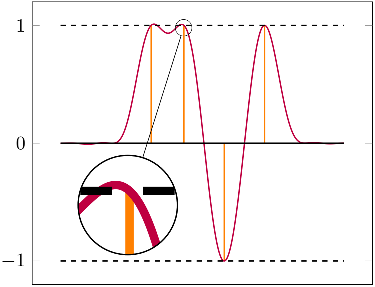


Problem: Magnitude of certificate locally exceeds 1

Certificate for super-resolution

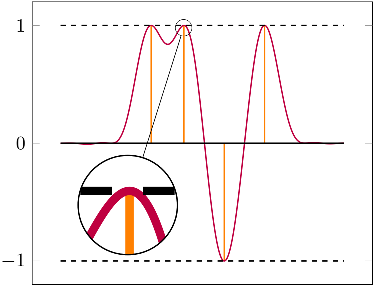


Problem: Magnitude of certificate locally exceeds 1

Solution: Add correction term and force the derivative of the certificate to equal zero on the support

$$c(t) = \sum_{i:x_i \neq 0} \alpha_i \, K(t-i) \; + \; \beta_i \, K'(t-i)$$

Certificate for super-resolution



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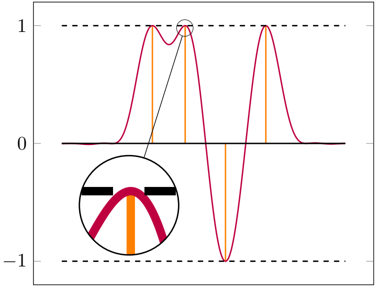
Certificate for super-resolution



Similar construction works for bandpass point-spread functions relevant to reflection seismology

Super-resolution from noisy data

Additive-noise model

$$y = \mathcal{F}_n x + z$$

Relaxed optimization problem

$$\min_{\tilde{x}} ||\tilde{x}||_{\text{TV}} \quad \text{subject to} \quad ||\mathcal{F}_n \tilde{x} - y||_2^2 \leq \delta$$

δ is an estimate of the noise level

Super-resolution from noisy data

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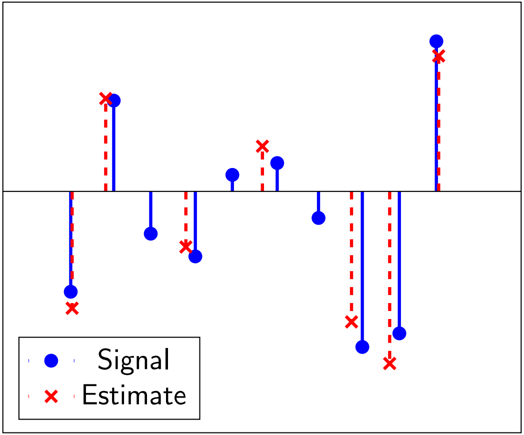
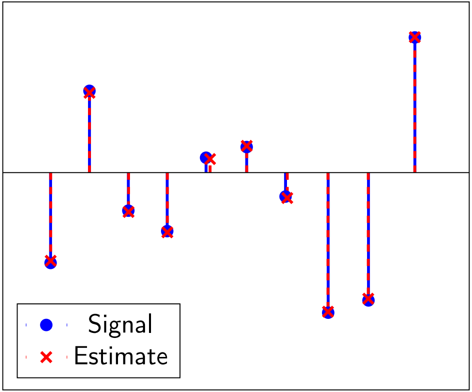
$\Delta = \frac{0.6}{f_c}$, SNR = 20 dB, $f_c = 40$



$\Delta = \frac{1.2}{f_c}$, SNR $= 20$ dB, $f_c = 40$



Deconvolution with the ℓ_1 norm (Taylor, Banks, McCoy '79)