Support Detection in Super-resolution

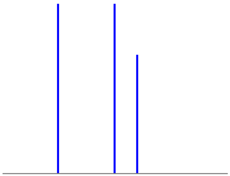
Carlos Fernandez-Granda (Stanford University)

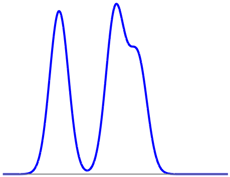
7/2/2013

Acknowledgements

This work was supported by a Fundación Caja Madrid Fellowship

Super-resolution





Aim : estimating fine-scale structure from low-resolution data

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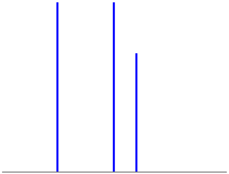

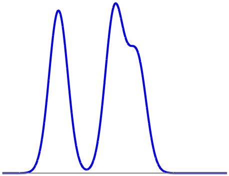

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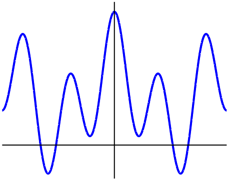
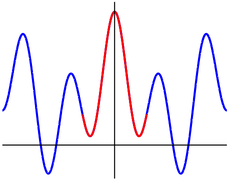
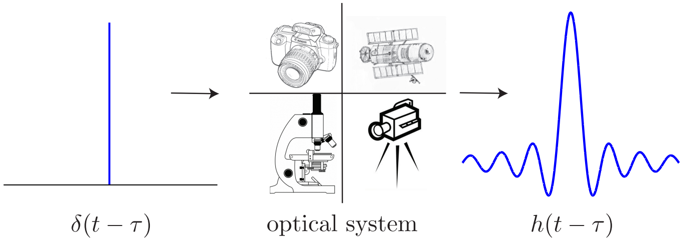




Equivalently, extrapolating the high end of the spectrum

Applications

Optics (diffraction-limited systems), electronic imaging, signal processing, radar, spectroscopy, medical imaging, astronomy, geophysics, etc.



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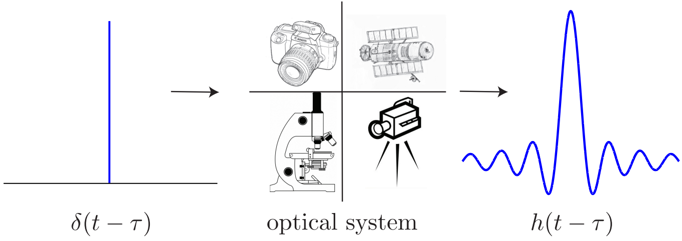

Signals of interest are often modeled as **superpositions of point sources**

- Celestial bodies in astronomy
- Line spectra in speech analysis
- Molecules in fluorescence microscopy

Mathematical model

- **Signal** : superposition of delta measures with support T

$$x = \sum_j a_j \delta_{t_j} \qquad a_j \in \mathbb{C},\ t_j \in T \subset [0,1]$$

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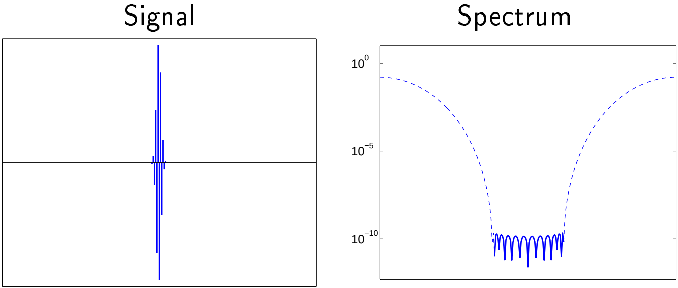
- **Measurement process** : low-pass filtering with cut-off frequency $\mathbf{f_c}$
- **Measurements** : $n = 2\,f_c + 1$ noisy low-pass Fourier coefficients

$$\begin{aligned} y(k) &= \int_0^1 e^{-i2\pi kt}\, x\,(\mathrm{d}t) + z(k) \\ &= \sum_j a_j e^{-i2\pi k t_j} + z(k), \quad k \in \mathbb{Z},\ |k| \leq f_c \\ y &= \mathcal{F}_n\, x + z \end{aligned}$$

When is super-resolution well posed ?



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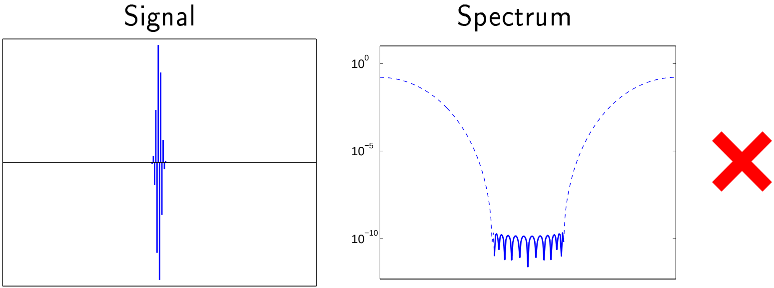

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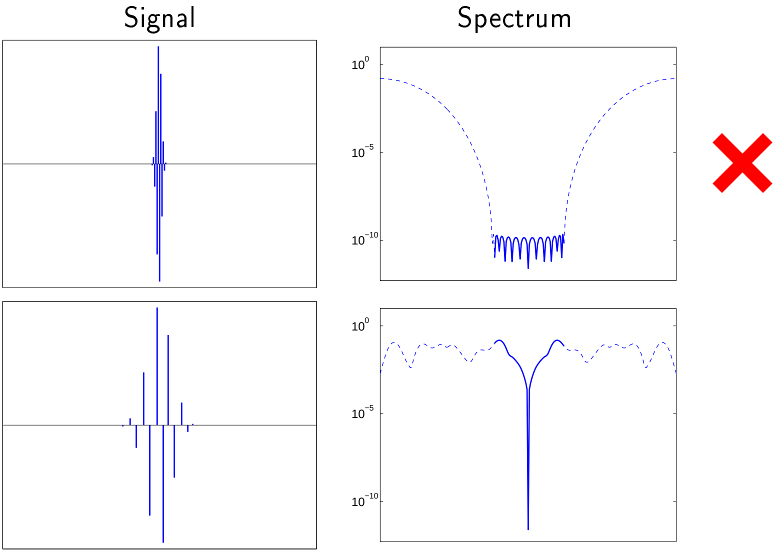

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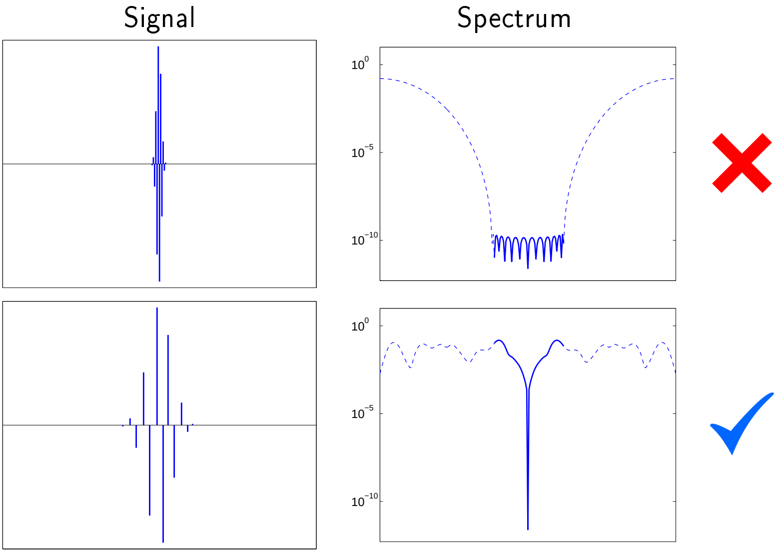

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- **Minimum separation** of the support T of a signal :

$$\Delta(T) = \inf_{(t,t') \in T \,:\, t \neq t'} |t - t'|$$

Total-variation norm

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- **Not** the total variation of a piecewise constant function
- **Formal definition :** For a complex measure ν

$$||\nu||_{\text{TV}} = \sup \sum_{j=1}^{\infty} |\nu(B_j)|,$$

(supremum over all finite partitions B_j of $[0,1]$)

Recovery via convex programming

In the absence of noise, i.e. if $y = \mathcal{F}_n \, x$, we solve

$$\min_{\tilde{x}} ||\tilde{x}||_{\mathrm{TV}} \quad \text{subject to} \quad \mathcal{F}_n \, \tilde{x} = y,$$

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Theorem [Candès, F. 2012]

If the minimum separation of the signal support T obeys

$$\Delta(T) \geq 2\,/f_c$$

then recovery is **exact**

Estimation from noisy data

- We consider noise bounded in ℓ_2 norm

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- The problem is infinite-dimensional, but its dual is **sdp-representable**

Implementation

Dual solution vector : Fourier coefficients of low-pass polynomial that *interpolates the sign of the primal solution*

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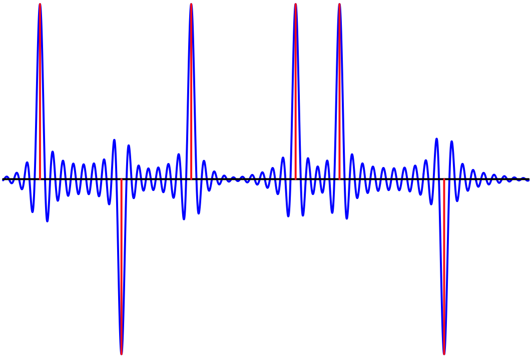

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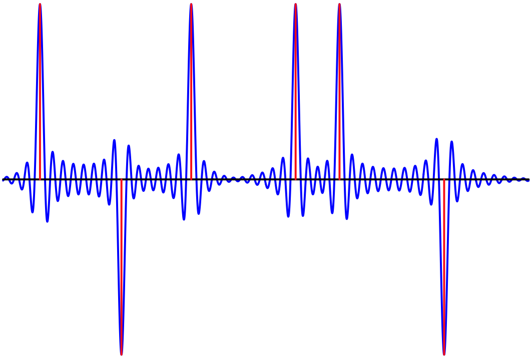

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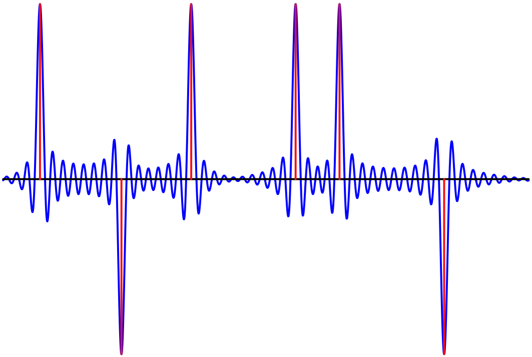

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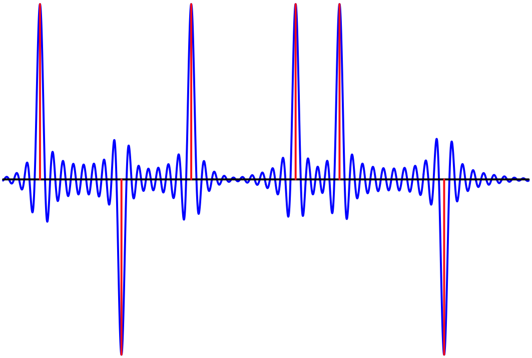
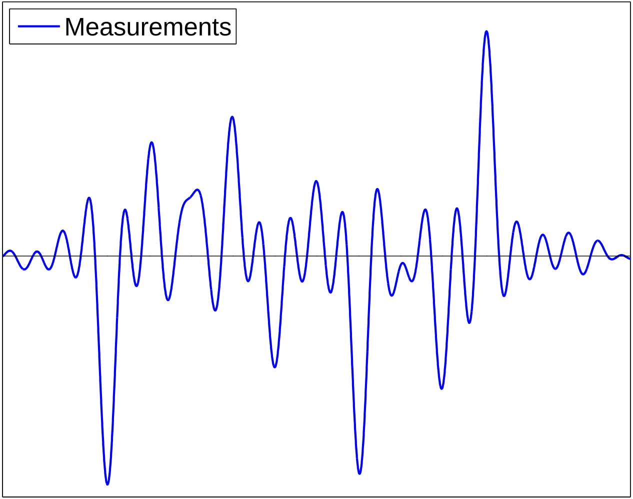


To estimate the support we

1. solve the sdp
2. determine where the magnitude of the polynomial equals 1

Example

SNR : 14 dB



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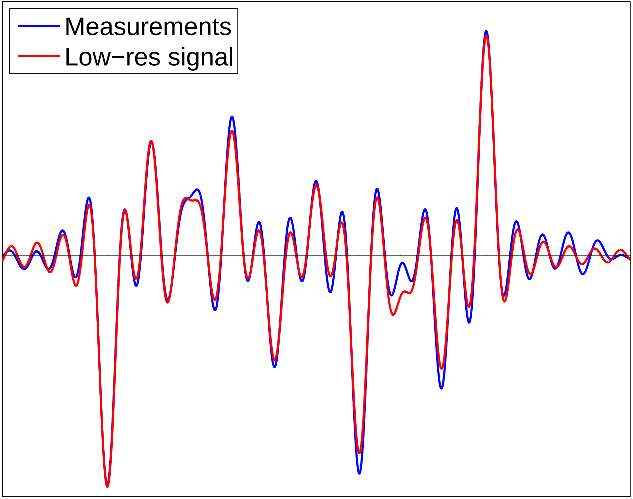

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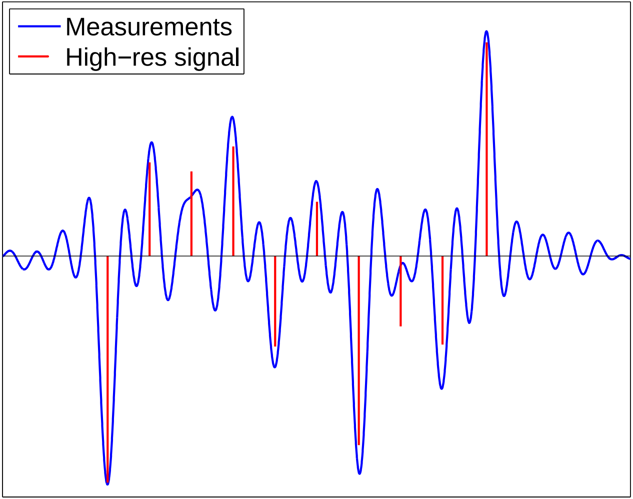
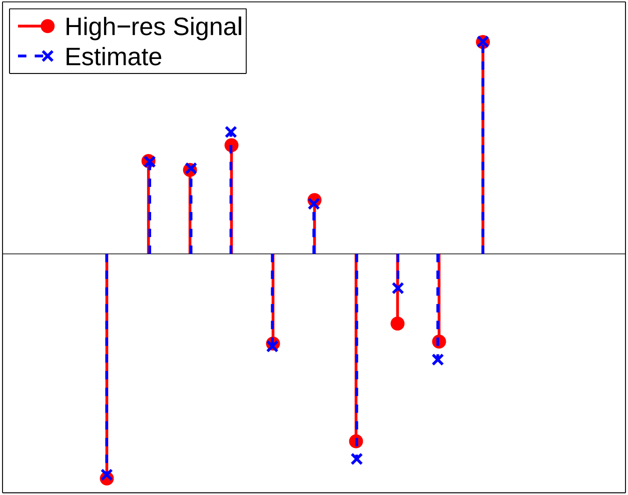


Example

Average localization error : $6.54\,10^{-4}$



Support-detection accuracy

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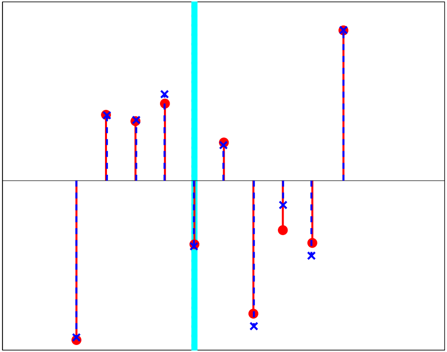
Main result : Theoretical guarantees on the quality of the estimate under the minimum-separation condition

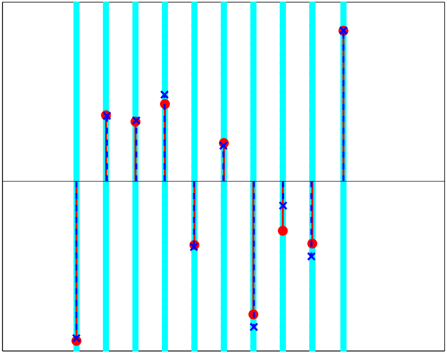
$$\Delta(T) \geq 2/f_c$$

Theorem [F. 2013] Spike detection



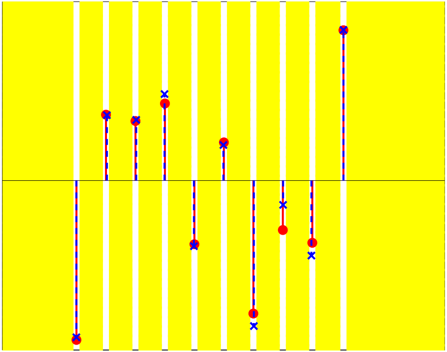
$$(1): \left| a_j - \sum_{\left\{\hat{t}_l \in \widehat{T} :\, \left|\hat{t}_l - t_j\right| \leq c/f_c\right\}} \hat{a}_l \right| \leq C_1 \delta \quad \forall t_j \in T, \qquad c := 0.1649$$

Theorem [F. 2013] Support-detection accuracy



$$(2): \sum_{\left\{\hat{t}_l \in \widehat{T},\, t_j \in T :\, \left|\hat{t}_l - t_j\right| \leq c/f_c\right\}} |\hat{a}_l| \left(\hat{t}_l - t_j\right)^2 \leq \frac{C_2 \delta}{f_c^2}, \quad c := 0.1649$$

Theorem [F. 2013] False positives



$$(3): \sum_{\left\{\hat{t}_l \in \widehat{T} : \left|\hat{t}_l - t_j\right| > c/f_c \, \forall t_j \in T\right\}} |\hat{a}_l| \leq C_3 \delta, \quad c := 0.1649$$

Support-detection accuracy

Corollary

For any $t_i \in T$, *if* $a_i > C_1\delta$ *there exists* $\hat{t}_i \in \hat{T}$ *such that*

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In previous work :

- The estimation errors of the different spikes cannot be decoupled [Candès, F. 2012]

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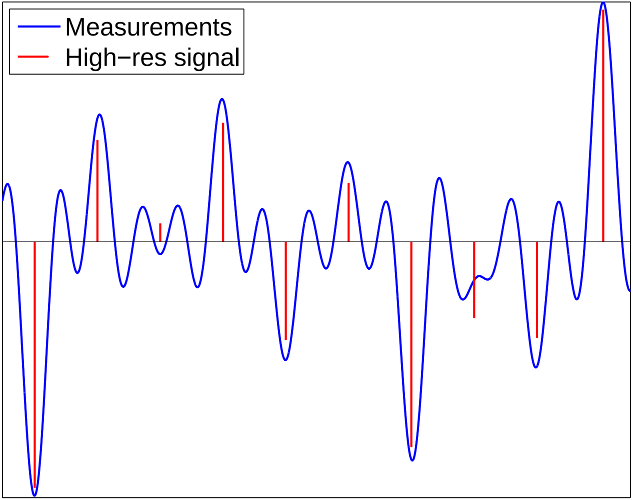
In previous work :

- The estimation errors of the different spikes cannot be decoupled [Candès, F. 2012]
- The bounds depend on the amplitude of the estimate, not of the original signal [Azais *et al* 2013]

Consequence

Robustness of the algorithm to high dynamic ranges



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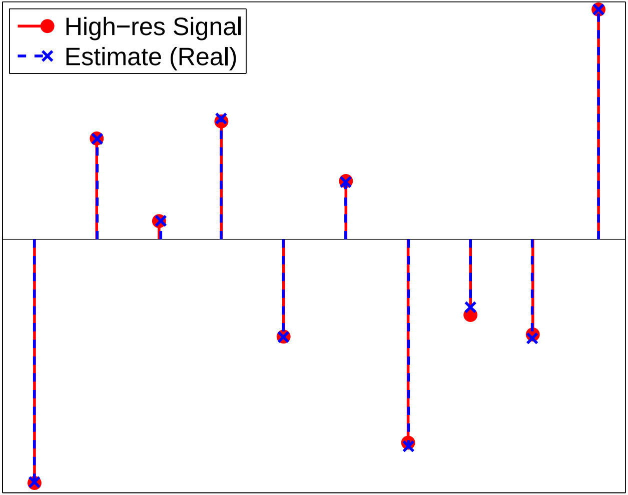

Proof techniques

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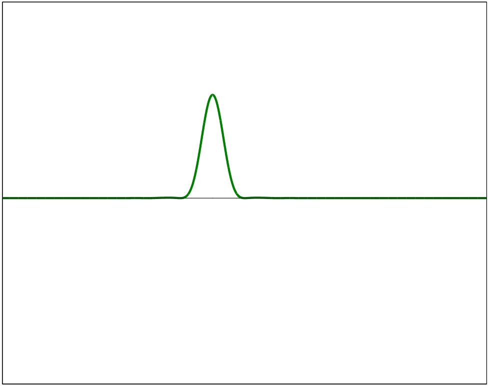
- To establish the **spike-detection** bound

$$\left| a_j - \sum_{\left\{\hat{t}_l \in \widehat{T}:\, \left|\hat{t}_l - t_j\right| \leq c/f_c\right\}} \hat{a}_l \right| \leq C_1 \delta \quad \forall t_j \in T$$

we construct a low-pass polynomial for each $t_j \in T$

$$q_{t_j}(t) = \sum_{k=-f_c}^{f_c} b_k e^{i2\pi kt}$$

Sketch of proof



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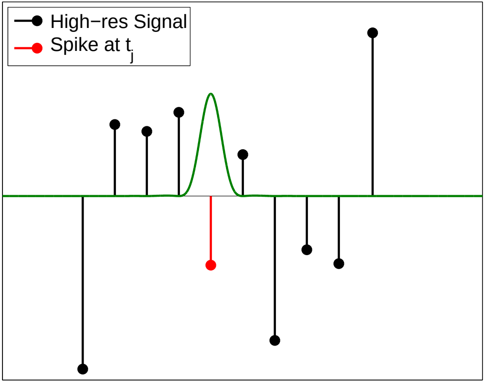

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$$\int_{[0,1]} q_{t_j}(t) x(\mathrm{d}t) = a_j$$

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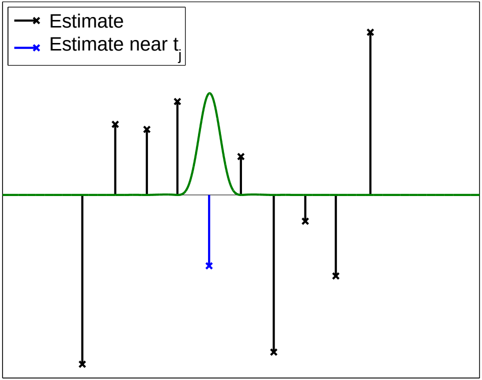

We can establish (using the support-detection bound)

$$\int_{[0,1]} q_{t_j}(t)\hat{x}(\mathrm{d}t) = \sum_{\left\{\hat{t}_l \in \widehat{T}:\, \left|\hat{t}_l - t_j\right| \leq c/f_c\right\}} \hat{a}_l + \Omega(\delta)$$

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$$\begin{aligned}
\left| a_j - \sum_{\left\{\hat{t}_l \in \widehat{T} : \left|\hat{t}_l - t_j\right| \leq \frac{c}{f_c}\right\}} \hat{a}_l \right| &= \left| \int_{[0,1]} q_{t_j}(t) x(\mathrm{d}t) - \int_{[0,1]} q_{t_j}(t) \hat{x}(\mathrm{d}t) \right| + \Omega(\delta) \\
&= \left| \sum_{k=-f_c}^{f_c} b_k \mathcal{F}_n (x - \hat{x})_k \right| + \Omega(\delta) \quad \text{(Parseval)} \\
&\leq \left|\left| q_{t_j} \right|\right|_{L_2} ||\mathcal{F}_n (x - \hat{x})||_2 + \Omega(\delta) \quad \text{(Cauchy-Schwarz)} \\
&= \Omega(\delta)
\end{aligned}$$

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- For more details,
  - C. Fernandez-Granda. *Support detection in super-resolution*. Proceedings of SampTA 2013
  - E. J. Candès and C. Fernandez-Granda. *Towards a mathematical theory of super-resolution*. To appear in Comm. on Pure and Applied Math.

Thank you