Itô Integral: Definition, Formulas and Models

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SDE in Modeling

In mathematical modeling, we will frequently encounter stochastic differential equations, i.e.:

$$dx_t = (-ax_t + \epsilon y_t + F_x(t))dt,$$

$$dy_t = (\epsilon^{-1}x_t - by_t + F_y(t))dt + \sigma dW_t.$$

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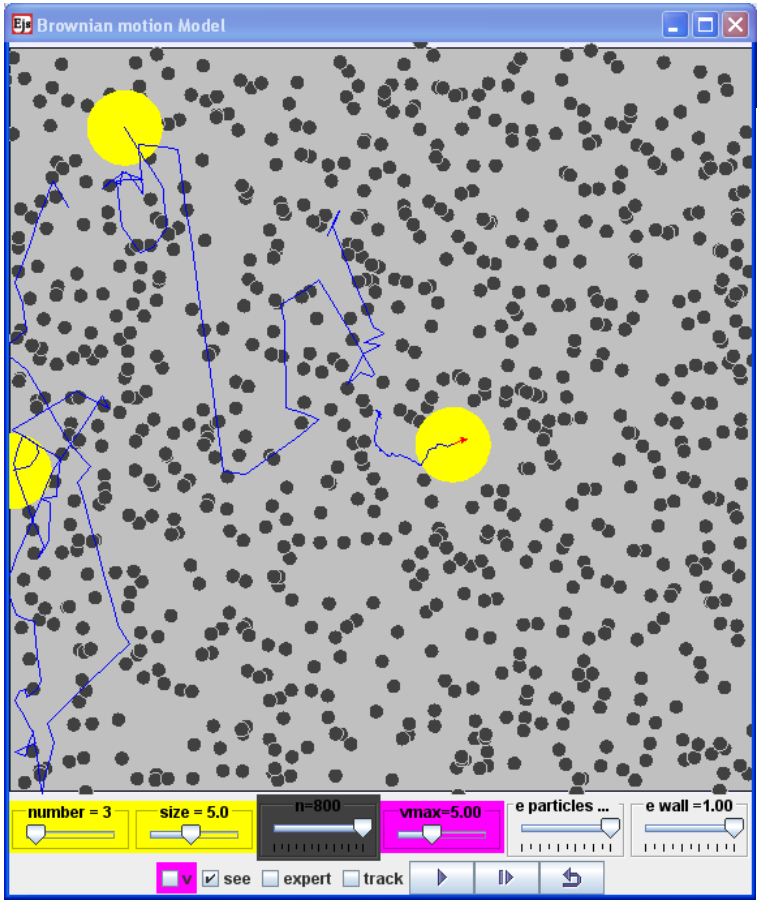
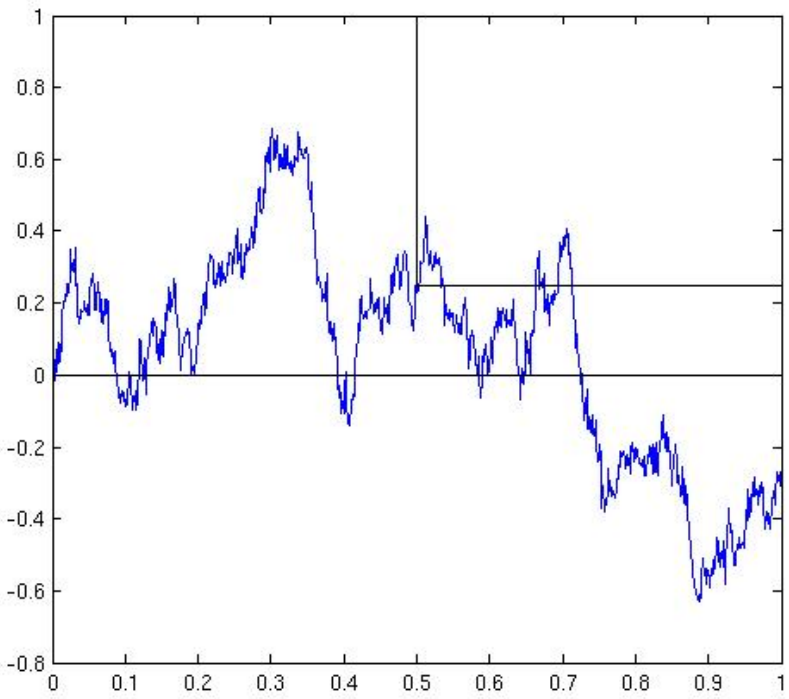
Brownian Motion

The Brownian motion, W_t, was first used as a mathematical description of random movement of particles suspended in a fluid.





Brownian Motion

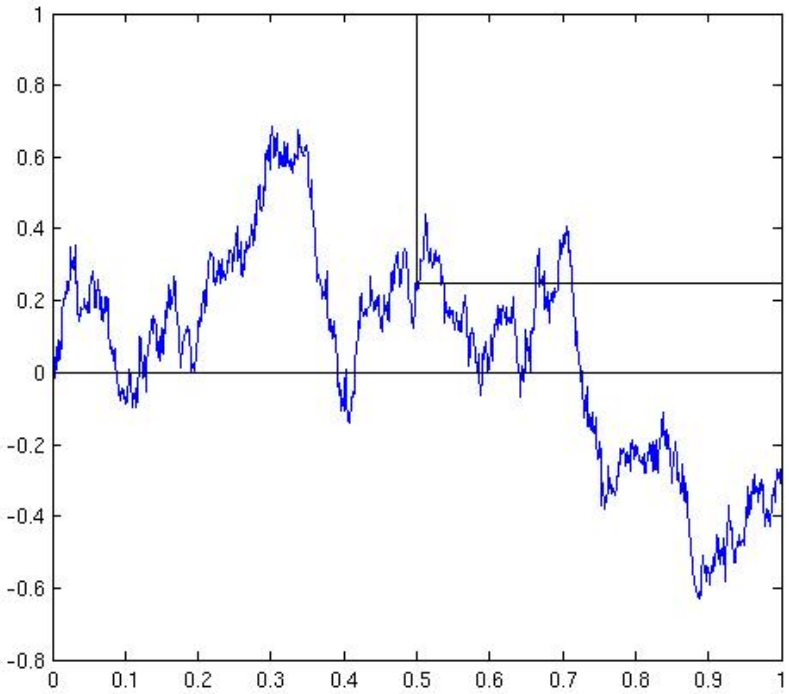
Intuitively, W_s should satisfies,

- $W_0 = 0$;
- $(W_s)_{s \le t}$ has continuous paths;
- Renormalization: $\mathbf{E}W_1^2 = 1$.
- Has stationary independent increments:

$$W_{t+s} - W_s \perp (W_r)_{r \le s},$$

$$W_{t+s} - W_s \sim W_t;$$

- Increments should be symmetric: $\mathbf{E}W_t = 0$;



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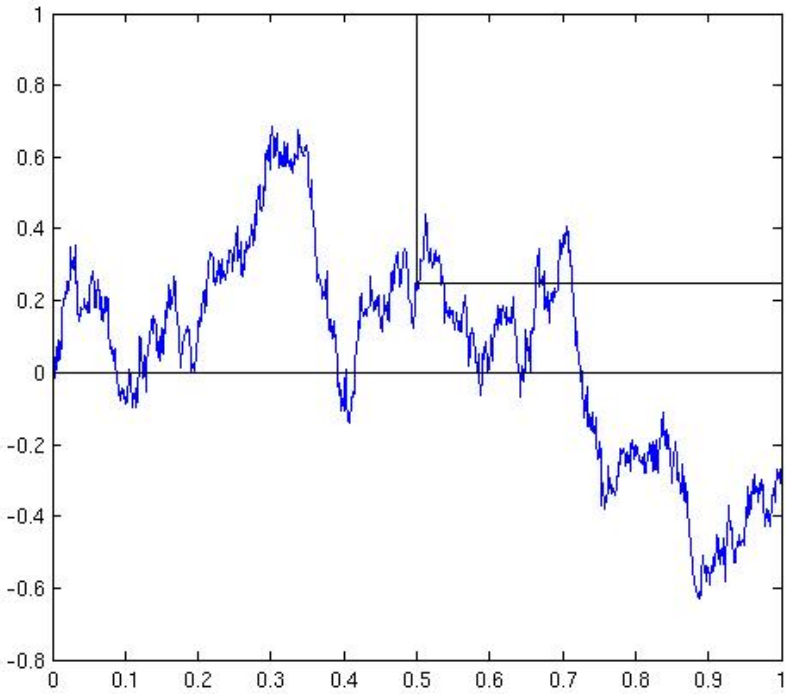

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Brownian Motion

A simpler definition

- $W_0 = 0$;
- $(W_s)_{s \le t}$ has continuous paths;
- Has independent increments that follows:

$$W_{t+s} - W_s \sim \mathcal{N}(0, t).$$

How do we get Gaussian distribution here? Central limit theorem!

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} Y_k \Rightarrow \mathcal{N}(0, \sigma^2).$$

$$W_t = \sum_{k=1}^{n} \left(W_{\frac{k}{n}} - W_{\frac{k-1}{n}} \right) \Rightarrow \text{Gaussian}.$$

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Itô Integral

We call $\int X_t dW_t$ an Itô integral if X_t is adapted (non-anticipating):

- You have the value of X_t at time t;
- Counterexample: $X_t = W_1$;
- Crucial fact needed for following derivation:

$$W_t - W_s \perp (X_r)_{r \leq s}, \quad s < t.$$

Itô Integral

Recall Riemann Integral:

- With a partition

$$0 = \pi_0 < \pi_1 < \ldots < \pi_n = t, \quad |\pi| := \max_k |\pi_{k+1} - \pi_k|.$$

- $$\lim_{|\pi| \to 0} \sum_{k=0}^{n-1} X_{t_k}(W_{\pi_{k+1}} - W_{\pi_k}), \quad t_k \in [\pi_k, \pi_{k+1}].$$

Itô Integral

Can we use the same definition again? Check $X_t = W_t$:

- $$\lim_{|\pi|\to 0}\sum_{k=0}^{n-1} X_{t_k}(W_{\pi_{k+1}} - W_{\pi_k}), \quad t_k \in [\pi_k, \pi_{k+1}]$$

$$\lim_{|\pi|\to 0}\sum_{k=0}^{n-1} W_{\pi_k}(W_{\pi_{k+1}} - W_{\pi_k}) \ \ (I), \quad \lim_{|\pi|\to 0}\sum_{k=0}^{n-1} W_{\pi_{k+1}}(W_{\pi_{k+1}} - W_{\pi_k}) \ \ (II)$$

- $\mathbf{E}[(II) - (I)]$

$$\mathbf{E}\lim_{|\pi|\to 0}\sum_{k=0}^{n-1}(W_{\pi_{k+1}} - W_{\pi_k})^2 = \lim_{|\pi|\to 0}\sum_{k=0}^{n-1}(\pi_{k+1} - \pi_k) = t$$

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dW_t

dW_t should not be interpret as common derivatives: W_t is not differentiable: let us see $(W_{t+\delta} - W_t)/\delta$ as $\delta \to 0$:

$$\mathbf{P}\left(\left|\frac{W_{t+\delta} - W_t}{\delta}\right| \leq M\right) = \mathbf{P}(|Z| \leq \sqrt{\delta} M) \to 0.$$

Itô Integral

- Itô integral: choose the left point,

$$\int_0^t X_s dW_s := \lim_{|\pi|\to 0} \sum_{k=0}^{n-1} X_{\pi_k}(W_{\pi_{k+1}} - W_{\pi_k})$$

- The limit is interpreted in $\mathbf{L}^2$ sense:

$$\mathbf{E}\left|S_\pi - \int_0^t X_s dW_s\right|^2 \to 0.$$

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Mean and Variance of Itô Integral

- $\mathbf{E}\int_0^t X_s dW_s = 0$:

$$\mathbf{E}\sum X_{\pi_k}(W_{\pi_{k+1}} - W_{\pi_k}) = 0 \overset{|\pi|\to 0}{\longrightarrow} 0;$$

- $\mathbf{E}\left(\int X_t dW_t\right)^2 = \int \mathbf{E}(X_t)^2 dt$:

$$\begin{aligned}
&\mathbf{E}\left(\sum X_{\pi_k}(W_{\pi_{k+1}} - W_{\pi_k})\right)^2 \\
&= \sum_k \mathbf{E}X_{\pi_k}^2 (W_{\pi_{k+1}} - W_{\pi_k})^2 \\
&\quad + 2\sum_{j<k} \mathbf{E}X_{\pi_k}X_{\pi_j}(W_{\pi_{j+1}} - W_{\pi_j}) \cdot (W_{\pi_{k+1}} - W_{\pi_k}).
\end{aligned}$$

Itô Formula

We call $X_t = \int_0^t a_s ds + \int_0^t b_s dW_s$ a Itô process. A simpler differential notation:

- $$dX_t = a_t dt + b_t dW_t$$

- Itô formula:

$$\begin{aligned} df(X_t) &= f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t \cdot dX_t) \\ &= f'(X_t)(a_t dt + b_t dW_t) + \frac{1}{2}f''(X_t)b_t^2 dt \end{aligned}$$

- New idea here:

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Itô formula

$$f(W_t) - f(0) = \int_0^t f'(W_s)dW_s + \frac{1}{2}f''(W_s)ds$$

Taylor expansion:

$$\sum_k f(W_{\pi_k}) - f(W_{\pi_{k-1}}) = \sum_k f'(W_{\pi_{k-1}})(W_{\pi_k} - W_{\pi_{k-1}}) + \frac{1}{2}\sum_k f''(W_{\pi_{k-1}})(W_{\pi_k} - W_{\pi_{k-1}})^2 + o$$

Comparing difference:

$$\sum_k f''(W_{\pi_{k-1}})(W_{\pi_k} - W_{\pi_{k-1}})^2 - \sum_k f''(W_{\pi_{k-1}})(\pi_k - \pi_{k-1}) = \sum_k f''(W_{\pi_{k-1}})[(W_{\pi_k} - W_{\pi_{k-1}})^2 - (\pi_k - \pi_{k-1})]$$

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Fokker Planck Equation

Assume $a_t = a_t(X_t), b_t = b_t(X_t)$, and write the density of X_t w.r.t. Lebesgue as p_t, then

$$\frac{\partial}{\partial t} p_t(x) = -\frac{\partial}{\partial x}(a_t(x)p_t(x)) + \frac{\partial^2}{\partial x^2}(b_t^2(x)p_t(x)).$$

Fokker Planck Equation

Take any good f, we have two ways to express $\mathbf{E}f(X_t)$:

$$\int_x p_t(x)f(x)dx = \mathbf{E}\int_0^t (f'(X_s)a_s(X_s)+\frac{1}{2}f''(X_s)b_s^2(X_s))ds+f'(X_s)b_s dW_s$$

Taking derivative w.r.t t on both hand:

$$\begin{aligned}\int \frac{\partial}{\partial_t} p_t(x)f(x)dx &= \mathbf{E}f'(X_t)a_t(X_t) + \frac{1}{2}f''(X_t)b_s^2(X_t) \\ &= \int_x (f'(x)a_t(x) + f''(x)b_t^2(x))p_t(x)dx \\ &= \int_x f(x)\Big[-\frac{\partial}{\partial x}(a_t(x)p_t(x)) + \frac{\partial^2}{\partial x^2}(b_t(x)p_t(x))\Big]dx.\end{aligned}$$

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Multidimensional Case

- From

$$dx_t = a_t dt + \sigma_1 dW_1(t) + \sigma_2 dW_2(t),$$

$$dy_t = b_t dt + \delta_1 dW_1(t) + \delta_2 dW_2(t).$$

- to

$$dX_t = A_t dt + \Sigma_t dW_t.$$

$$X_t = \begin{bmatrix} x_t \\ y_t \end{bmatrix}, \quad A_t = \begin{bmatrix} a_t \\ b_t \end{bmatrix}, \quad \Sigma_t = \begin{bmatrix} \sigma_1, & \sigma_2 \\ \delta_1, & \delta_2 \end{bmatrix}, \quad W_t = \begin{bmatrix} W_1(t) \\ W_2(t) \end{bmatrix}$$

- Itô formula: $dW_t dW_t^T = I dt$.

SDE

- Gaussian process:

$$X_t = x_0 + \int_0^t a_{r,t} dr + \int_0^t b_{r,t} dW_r$$

$$cov(X_t, X_s) = \int_0^{s \wedge t} b_{r,t} b_{r,s} dr.$$

- We want to solve

$$dX_t = (AX_t + F_t)dt + B_t dW_t.$$

SDE solution

- ODE solution for $dx_t = (ax_t + f_t)dt$:

$$d(e^{-at}x_t) = -ae^{-at}x_t dt + e^{-at}dx_t = e^{-at}f_t dt$$

- Likewise for $dx_t = (ax_t + f_t)dt + b_t dW_t$:

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- Solution:

$$X_t = e^{at}x_0 + \int_0^t e^{a(t-s)}f_s ds + \int_0^t e^{a(t-s)}b_s dW_s.$$

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Matrix Exponential

- $\exp(At) = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots;$
- $\frac{\partial}{\partial_t}\exp(At) = \exp(At)A = A\exp(At);$
- Real computation needs Jordan form: $A = PJP^{-1}$

$$J = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & \dots \\ 0 & \lambda_2 & 1 & 0 & \dots \\ 0 & 0 & \lambda_2 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}.$$

- $$J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \Rightarrow \exp(At) = P \begin{bmatrix} \exp(\lambda_1 t) & 0 \\ 0 & \exp(\lambda_2 t) \end{bmatrix} P^{-1}$$

Linear Two Levels Model

Consider a two level process

$$dx_t = (-ax_t + \epsilon y_t + F_x(t))dt,$$

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Linear Two Levels Model

Assumes A has distinct eigenvalues λ_1, λ_2,

$$A = P \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} P^{-1} \quad \Rightarrow \quad e^{At} = P \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} P^{-1}.$$

For X_t has equilibrium state,

$$X_t = e^{At} x_0 + \int_0^t e^{A(t-s)} F_s ds + \int_0^t e^{A(t-s)} B dW_s,$$

both λ_1, λ_2 should have negative real parts:

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Asymptotic Statistics

- Time shift:

$$\lim_{t\to\infty}\int_0^t e^{A(t-s)}F_s ds = \lim_{t\to-\infty}\int_{-t}^0 e^{As}F_{t+s}ds$$

- Assume $F_s \equiv F$, and care only about asymptotic

$$X_t = \int_{-\infty}^0 e^{As}Fds + \int_{-\infty}^0 e^{As}BdW_s$$

$$X_{t+r} = \int_{-\infty}^r e^{A(r+s)}Fds + \int_{-\infty}^r e^{A(r+s)}BdW_s$$

- $\mathbf{E}X_t = P\begin{bmatrix}-1/\lambda_1, & 0\\ 0, & -1/\lambda_2\end{bmatrix}P^{-1}F = -A^{-1}F$

$$cov(X_t, X_{t+r}) = \int_{-\infty}^0 e^{A(r+s)}B^2e^{As}ds.$$

Marginal Statistics

- Recall that we want to compare marginal statistics of $x_t = X_t^{(1)}$ with the following process:

$$du_M(t) = -\gamma_M u_M(t) + F_M + \sigma_M dW_t.$$

- The distribution of u_M can be fitted to the one of x_t exactly.
- $$\mathbf{E}x_\infty = \frac{bF_x + \epsilon F_y}{ab-1}, \quad var(x_\infty) = \frac{\sigma^2\epsilon^2}{2(a+b)(ab-1)}$$

$$\mathbf{E}u_M(\infty) = \frac{F_M}{\gamma_M}, \quad var(u_M(\infty)) = \frac{\sigma_M^2}{2\gamma_M}$$