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Office hours: TBD

Weekly homework, to begin next week

Recitation: Fri 3:30 - 4:45 CIWW 201

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Commun.
Piazza?
Campuswire?

Grading: HW 10% (weekly)

Exam 1 25% Exam 2 25%

Final 40%

Overall numerical => Letter (Any curve will only help you.)

Grads: NYU classes

Cheating policy: HW → problem gets 0

Exam → exam taken, 0, referred to Dept. Administrator for adjudication

Comments

- Best way to study for exams is to do the homework.
- No late homework accepted without prior permission and valid excuse (e.g. medical, etc.)
- Same policy for exams, prior approval required.

Prerequisites: - Calc III

- Linear algebra
  - ↳ who remembers eigenvalues?
    - → null space?
    - → vector space?
    - → ?

Overview:

Algebra: Solve $3x^2 + 2x - 1 = 5$

- coefficients are numbers, unknown is a number.

Calculus: Compute derivatives and integrals

$$\frac{d}{dx}\left(5xe^x\right) = \ldots$$

$$\int 5xe^x \, dx = \ldots$$

Linear Algebra Solve $A\vec{x} = \vec{b}$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

ODEs Algebra + Calculus + Linear Algebra

For example, one goal of the course will be to solve the equation:

$$u''(x) + p(x)\,u'(x) + q(x)\,u(x) = f(x)$$

for the function $u = u(x)$.

First order equations (Braun Ch. 1)

Ex:

$$\boxed{\frac{dy}{dt}} = \cos y + e^{t} - yt$$

First order

$$\boxed{\frac{d^3y}{dt^3}} - \frac{d^2y}{dt^2} + \cos t + y = 0.$$

Third order

The order of the differential equation (DE) is the order of the highest derivative.

The solution is the function y, that along with its derivatives satisfies the equation.

Applications of DE are <u>everywhere</u> in math & science:

- at their core, ODE's merely provide a description of how one quantity (y) changes over time (t).

- Contrast with <u>Partial</u> DE, which describe how a quantity (u) changes with regard to several variables (t, x, y)

Ex: $$\underbrace{\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}_{\text{Toward end of course, or MATH 263.}}$$ partial derivatives

<u>1st order equations</u>

(Begin with examples, calculus manipulations, return later to existence & uniqueness)

General form:

$$(*) \qquad \frac{dy}{dt} = \underbrace{f(y,t)}$$

a function of the function y and t: Ex: $y(t)^2 + t$

Most of the time, $(*)$ cannot be solved without the help of a computer.

If however we have

$$\frac{dy}{dt} = g(t),$$

then both sides can be integrated:

$$\int \frac{dy}{dt}\, dt = \underbrace{\int g(t)\, dt}_{\text{anti-derivative of } g} + C$$

CF
- Anti derivative
- Integral
- FTC

$$\int \frac{dy}{dt}\, dt = \int dy = y$$

$$\Rightarrow \quad y(t) = \int g(t)\, dt + c$$

c is a constant that can be determined from an initial condition

Ex: $y(a) = b$.

Ex:

$$\frac{dy}{dt} = t^3 + \cos t$$

$$y(0) = 1$$

$$\Rightarrow \quad y(t) = \int (t^3 + \cos t)\, dt \quad + c$$

$$= \frac{1}{4}t^4 + \sin t + c$$

$$y(0) = 0 = 1 \quad \Rightarrow \quad c = 1$$

$$\Rightarrow \quad y(t) = \frac{1}{4}t^4 + \sin t + 1.$$

Linear vs. Nonlinear Equations

Linear: $\quad y' + a(t)\, y = b(t)$

Nonlinear: $\quad y' + yy' + c(t)\, y^2 = d(t)$

Linear equations are of the form $Ly = f$, where L is a linear differential operator:

$$L = \frac{d}{dt} + a(t)$$

$$\Rightarrow \quad L(c_1 y_1 + c_2 y_2) = \frac{d}{dt}(c_1 y_1 + c_2 y_2) + a(t)(c_1 y_1 + c_2 y_2)$$

$$= c_1 y_1' + a(t) c_1 y_1 + c_2 y_2' + a(t) c_2 y_2$$

$$= c_1 L y_1 + c_2 L y_2$$

(Recall linear operator/transform from linear alg.)

Lin. 1st order equations

$$Ly = b \Rightarrow \frac{dy}{dt} + a(t)\,y = b(t).$$

Assume $b=0$ for now. Then, move "y terms" to left, "t" terms to right:

$Ly = b$ Nonhomogeneous / Inhomogeneous

$Ly = 0$ Homogeneous

$$\frac{y'}{y} = -a(t)$$

Note now that $\frac{d}{dt}\log|y| = \underbrace{\frac{y'}{y}}_{\text{logarithmic derivative}}$

natural logarithm

$$\Rightarrow \frac{d}{dt}\log|y| = -a(t) \quad \text{can be integrated}$$

$$\int \frac{d}{dt}\log|y|\,dt = \int -a(t)\,dt + c_1$$

$$\log|y| = \int -a(t)\,dt + c_1$$

Exponentiate both sides:

$$|y| = e^{-\int a(t)\,dt + c_1} = c\,e^{-\int a(t)\,dt} \quad \text{with } c = e^{c_1}$$

What about these absolute value signs?

We have that $\underbrace{\left|y\,e^{\int a(t)\,dt}\right|}_{\text{continuous function}} = c$, so either $y e^{\int a} = \pm c$.

Since c is arbitrary, $\underbrace{y = c\,e^{-\int a(t)\,dt}}_{\text{General solution to the homogeneous equation.}}$

Ex: $y' + \cos t \, y = 0$

$\Rightarrow \frac{y'}{y} = -\cos t \quad \Rightarrow \frac{d}{dt} \log|y| = -\cos t$

$\Rightarrow y = c\, e^{-\int \cos t \, dt} = c\, e^{-\sin t}$. c can be determined from data $y(t_0) = y_0$.

Ex: $y' + a(t)\, y = 0$

$y(t_0) = y_0$

$\Rightarrow \frac{d}{dt} \log|y| = -a(t)$

$$\int_{t_0}^{t} \frac{d}{d\tau} \log|y(\tau)| \, d\tau = -\int_{t_0}^{t} a(\tau)\, d\tau$$

$$\log|y(t)| - \log|\underbrace{y(t_0)}_{y_0}| = -\int_{t_0}^{t} a(\tau)\, d\tau$$

$$\Rightarrow y(t) = y_0\, e^{\underbrace{-\int_{t_0}^{t} a(\tau)\, d\tau}}$$

In general, this must be computed numerically, as most functions do not have closed form anti-derivatives.

Note: when $t \to t_0$,

$\int_{t_0}^{t} a(\tau)\, d\tau \to 0$,

$\Rightarrow e^{-\int a} \to 1$

$\Rightarrow y(t) \to y(t_0) = y_0$.