Sep 9, 2019

Announcements:

HW 1 posted to website after class cims.nyu.edu/~oneil/ode

Recitation instructor will soon set office hours.

Due Mon, Sep 16 at the start of class. No late homework.

Last time:

Classification of ODEs:

- Linear vs. nonlinear
- Homogeneous vs. inhomogeneous
  ($y' + a(t) y = 0$ $y' + a(t) y = b(t)$)
- Order of ODE is highest derivative appearing

Example

$$y' + \cos t \; y = 0$$

$$y(0) = 1$$

$$\Rightarrow \frac{y'}{y} = -\cos t \quad \Rightarrow \frac{d}{dt} \log |y| = -\cos t$$

Integrate: $\displaystyle\int_0^t \frac{d}{d\tau} \log |y(\tau)| \, d\tau = -\int_0^t \cos \tau \, d\tau = -\sin t$

$$\log \frac{|y(t)|}{\underbrace{|y(0)|}_{=1}} = -\sin t$$

$$\Rightarrow y(t) = e^{-\sin t}$$

Solving Inhomogeneous DEs

$$Ly = b \quad \Rightarrow \quad \underset{(**)}{y' + ay = b.}$$

Can we use a similar integrating idea? ($\frac{d}{dt}(?) = b$, then integrate?)

Use an <u>integrating factor</u> $\mu = \mu(t)$.

Multiply $(**)$ by μ:

$$\mu y' + a\mu y = \mu b.$$

Can μ be chosen such that $\mu y' + a\mu y = \frac{d}{dt}(\mu y)$?

$(\mu y)' = \mu y' + \mu' y$, and therefore we must choose μ such that $\mu' = a\mu$.

This is easily solved, as above, to be $\boxed{\mu = e^{\int a(t)\,dt}}$

μ is called the integrating factor.

With this choice, we have:

$$\mu y' + a\mu y = b$$

$$\Rightarrow \frac{d}{dt}(\mu y) = b \quad \text{integrate...}$$

$$\mu y = \int b(t)\,dt + c$$

$$\Rightarrow y = \frac{1}{\mu}\left(\int b(t)\,dt + c\right) = e^{-\int a(t)\,dt}\left(\int b(t)\,dt + c\right)$$

Example

$$y' + 2ty = t \quad , \quad y(1) = 2$$

Integrating factor : $\mu(t) = e^{\int 2t\, dt} = e^{t^2}$

$$\Rightarrow \quad e^{t^2} y' + 2t e^{t^2} y = t e^{t^2}$$

$$\left(e^{t^2} y \right)' = t e^{t^2}$$

$$\int_1^t e^{\tau^2} y(\tau)\, d\tau = \int_1^t \tau e^{\tau^2}\, d\tau$$

$$e^{t^2} y(t) - 2e = \frac{1}{2} e^{\tau^2} \Big|_1^t = \frac{e^{t^2}}{2} - \frac{e}{2}$$

$$\Rightarrow \quad y(t) = \frac{1}{2} + \frac{3}{2} e^{1-t^2}$$

Application : Carbon dating

Details of the "Vermeer forgeries" in the text book,
short story is that different pigments were found in the paints
and this was verified using CARBON DATING.

Curies/ Rutherford: radioactive elements decay into others. (randomly)

Example: Carbon

Carbon-12	:	6 protons, 6 neutrons	99% of carbon
Carbon-13	:	7 neutrons,	1%
Carbon-14	:	8 neutrons	1-1.5 atoms per 10^{12} carbon atoms

Decay: $C_{14} \to N_{14}$ + energy

Nitrogen : 7 p, 7 N

"Half life" of C_{14} is 5730 years, meaning "half" of atoms have turned into N_{14}. If the original number was known, the substance could be dated.

Clearly the rate of N_{14} produced is proportional to original number of C_{14}: $N' = -\lambda N$

N = number of C_{14}. $N > 0$, $N' \leq 0$ always.

With initial condition $N(t_0) = N_0$ we can solve this ODE:

$$\int_{t_0}^{t} \frac{N'(\tau)}{N(\tau)}\, d\tau = \int_{t_0}^{t} -\lambda \, d\tau = -\lambda(t - t_0)$$

$$\Rightarrow \quad N(\tau) = N_0 e^{-\lambda(t-t_0)} \qquad \text{(from earlier)}$$

If we are interested in when $\frac{N}{N_0} = \frac{1}{2}$, then we can solve:

$$\frac{1}{2} = e^{-\lambda(t-t_0)}$$

$$\log\frac{1}{2} = -\lambda(t-t_0) \quad \Rightarrow \quad t - t_0 = -\frac{1}{\lambda}\log\frac{1}{2} = \frac{\log 2}{\lambda}$$

$$\Rightarrow \quad \text{half-life } t_H = \frac{\log 2}{\lambda}$$

Ex: C_{14}: $t_H = 5730$

U_{238}: 4.5 BILLION YRS

U_{235}: 700 MILLION YRS

Different types of decay, different risks...

So, in order to DATE a substance, first compute $\frac{N_0}{N}$ ← estimate from chemistry; N ← measure.

then the age of the substance:

$$t - t_0 = -\frac{1}{\lambda}\log\frac{N}{N_0}$$

See text for detailed chain of radioactive decay patterns, starting with U_{238}.

In summary: Homogeneous equation : $y' + a(t) y = 0$

$$\Rightarrow \frac{y'}{y} = -a(t)$$ direct integration

Inhomogeneous equation : $y' + a(t) y = b(t)$

$\Rightarrow$ use integrating factor to rewrite

In each case, we separated these equation.

Any ODE of the form

$$\frac{dy}{dt} = \frac{g(t)}{f(y)}$$ is called separable.

This can be written as:

$$f(y)\, dy/dt = g(t)$$

Computing the anti-derivation of both sids we have

$$\int f(y)\, dy/dt \; dt = \int g(t)\, dt + c$$

$$\underbrace{\int f(y)\, dy}_{= F(y)} = \int g(t)\, dt + c$$

$= F(y)$, when $F'(y) = f(y)$.

Once $F(y)$ is computed, the equation

$F(y) = \int g(t)\, dt + C$ must be solved for y.

Example: $\frac{dy}{dt} = t^2/y^3$

$$\Rightarrow \quad y^3 \frac{dy}{dt} = t^2 \quad \Rightarrow \quad \int y^3\, dy = \int t^2\, dt + c$$

$$\frac{1}{4}y^4 = \frac{1}{3}t^3 + c$$

$$y = \left(\frac{4}{3}t^3 + c\right)^{1/4}$$

Incorporating initial conditions is exactly the same as before:

Ex: $\frac{dy}{dt} = t^2/y^3$, $y(1) = 1$

Option 1: insert and solve for c in $y = \left(\frac{4}{3}t^3 + c\right)^{1/4}$

$$\Rightarrow 1 = \frac{4}{3} + c \Rightarrow c = -\frac{1}{3}$$

Option 2: Insert limits of integration

$$\int_1^y w^3\, dw = \int_1^t \tau^2\, d\tau$$

$$\frac{1}{4}w^4\Big|_1^y = \frac{1}{3}\tau^3\Big|_1^t \Rightarrow \frac{1}{4}y^4 - \frac{1}{4} = \frac{1}{3}t^3 - \frac{1}{3}$$

$$\Rightarrow y^4 = \frac{4}{3}t^3 - \frac{4}{3} + 1$$

$$\Rightarrow y = \left(\frac{4}{3}t^3 - \frac{1}{3}\right)^{1/4}$$

Let's examine another example where something goes wrong:

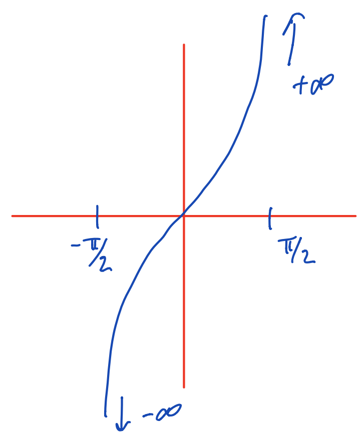
Ex: $y' = 1 + y^2$, $y(0) = 0$

$$\Rightarrow \int_0^y \frac{1}{1+w^2}\, dw = \int_0^t d\tau$$

$$\operatorname{atan} y = t$$

$$\Rightarrow y = \tan t.$$

Plot



What is the problem? The solution cannot be smoothly extended outside the interval $\underbrace{(-\pi/2, \pi/2)}$. This is the case most of the time. This interval is called the interval of existence.

(we will formalize this concept later on)