Sept 23, 2019

Last time:

Existence & Uniqueness

<u>Main theorem</u>: (Thm 2' in text book, §1.10)

Let f and $\partial f/\partial y$ be continuous in $\underbrace{t_0 \le t \le t_0 + a,\ |y - y_0| \le b}_{\text{Rectangle } R}$.

Compute $M = \max_R |f(t,y)|$ and set $\alpha = \min(a, b/M)$.

Then the IVP $y'(t) = f(t,y)$, $y(t_0) = y_0$ has a unique solution on the interval $t_0 \le t \le t_0 + \alpha$. (See textbook for detailed proof using Picard Iterations.)

<u>Example</u>: (Failure of uniqueness)

$$y' = \sqrt{y}, \quad y(0) = 0$$

Clearly $y(t) = 0$ is a solution.

Also: $\frac{y'}{\sqrt{y}} = 1 \Rightarrow \int_0^y \frac{1}{\sqrt{w}}\,dw = \int_0^t dt$

$\Rightarrow 2\sqrt{y} = t \Rightarrow \boxed{y = \frac{t^2}{4}}$ Also a solution $y' = \frac{t}{2} = \sqrt{y}$.

Checking the conditions of the Thm:

$f(t,y) = \sqrt{y}$ is continuous everywhere

$\frac{\partial f}{\partial y} = \frac{1}{2}\frac{1}{\sqrt{y}}$ is <u>not</u> continuous at the initial value, $(0,0)$.

<u>Next up</u>: - Numerical Methods for Solving IVPs.

As mentioned, very rarely is the IVP

$$y'(t) = f(t,y), \quad y(t_0) = y_0$$

solvable analytically.

The most simple approximation a some point $t > t_0$, but <u>close</u>, is to use the Taylor series approximation:

$$y(t) \approx y(t_0) + y'(t_0)(t - t_0)$$

Using the information available, we have:

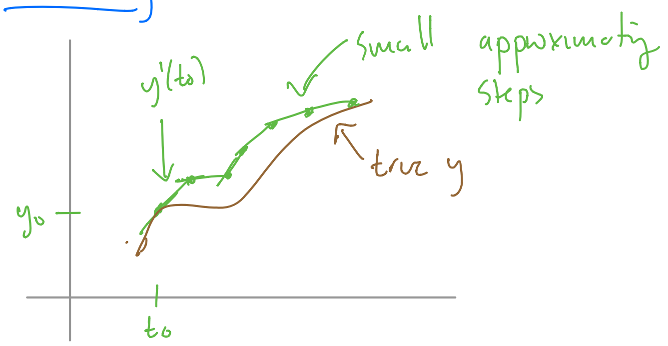
$$y(t) \approx y(t_0) + f(t_0, y_0)(t - t_0)$$

If we look at $t_0, t_1, \ldots$ and $t_{\ell+1} - t_\ell = h$, then

$$y(t_1) \approx y(t_0) + h f(t_0, y_0)$$ ← this is known as

EXPLICIT EULER Method

Pictorially:



To summarize:

$$y_\ell \approx y(t_\ell)$$

Define $$y_{\ell+1} = y_\ell + h \underbrace{f(t_\ell, y_\ell)}_{\text{approximation to } y'(t_\ell).}$$

(since we do not know the true y_ℓ).

Example

$$y'(t) = 1 + (y - t)^2$$
$$y(t_0) = y_0$$

Explicit Euler: $$y_{\ell+1} = y_\ell + \left(1 + (y_\ell - t_\ell)^2\right) h$$

Error Analysis

Recall the Taylor series:

$$y(t) = y(t_0) + y'(t_0)(t - t_0) + \frac{y''(t_0)}{2!}(t - t_0)^2 + \ldots$$

Taylor's Theorem says that if we truncate this, then

$$y(t) \underset{\text{EQUALS}}{=} y(t_0) + y'(t_0)(t - t_0) + \frac{y''(\xi)}{2!}(t - t_0)^2$$

ξ is some number in the interval $[t_0, t]$.

To find the error in Euler's Method we examine $\underbrace{y_{\ell+1}}_{\text{approx}} - \underbrace{y(t_{\ell+1})}_{\text{true value}}$

Euler: $y_{\ell+1} = y_\ell + h f(t_\ell, y_\ell)$

Taylor: $y(t_{\ell+1}) = y(t_\ell) + y'(t_\ell) h + \dfrac{y''(\xi_\ell)}{2!} h^2$

$$y_{\ell+1} - y(t_{\ell+1}) = y_\ell - y(t_\ell) + h\left(f(t_\ell, y_\ell) - f(t_\ell, y(t_\ell))\right) - \frac{y''(\xi_\ell)}{2} h^2$$

Note that $f(t_\ell, y_\ell) - f(t_\ell, y(t_\ell)) = \underbrace{\dfrac{f(t_\ell, y_\ell) - f(t_\ell, y(t_\ell))}{y_\ell - y(t_\ell)}}_{= \frac{\partial f}{\partial y}(t_\ell, \eta_\ell)} (y_\ell - y(t_\ell))$

some η_ℓ

$$\Rightarrow |y_{\ell+1} - y(t_{\ell+1})| \le |y_\ell - y(t_\ell)| + \left|\frac{\partial f}{\partial y}(t_\ell, \eta_\ell)\right| |y_\ell - y(t_\ell)| h + \frac{|y''(\xi_\ell)|}{2} h^2$$

Set $\epsilon_\ell = |y_\ell - y(t_\ell)|$ (error on step ℓ)

$$\Rightarrow \epsilon_{\ell+1} \le \epsilon_\ell + \left|\frac{\partial f}{\partial y}(t_\ell, \eta_\ell)\right| \epsilon_\ell h + \frac{|y''(\xi_\ell)|}{2} h^2$$

$$= \left(1 + h\left|\frac{\partial f}{\partial y}(t_\ell, y_\ell)\right|\right) \epsilon_\ell + \frac{|y''(\xi_\ell)|}{2} h^2$$

$$\le (1 + hL) \epsilon_\ell + \frac{D}{2} h^2$$

with $L = \max \left|\frac{\partial f}{\partial y}\right|$

$D = \max |y''|$ and note $y'' = \frac{d}{dt} y'$

$$= \frac{d}{dt} f(t,y) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f$$

To summarize:

$$\epsilon_{\ell+1} \le (1 + hL) \epsilon_\ell + \frac{D}{2} h^2$$

Can we say anything about ϵ_ℓ independent of $\epsilon_{\ell-1}$?

If $\epsilon_{k+1} \le A\epsilon_k + B$, then we can show that

$$\epsilon_k \le \frac{B}{A-1}\left(A^k - 1\right)$$

$$= \frac{D}{2}h^2 \frac{1}{1+hL-1}\left((1+hL)^k - 1\right)$$

$$= \underset{\substack{\downarrow \\ \to 0 \text{ as } h \to 0}}{\frac{D}{2L}h}\underbrace{\left((1+hL)^k - 1\right)}_{\to 0 \text{ as } h\to 0}$$

After some more algebra

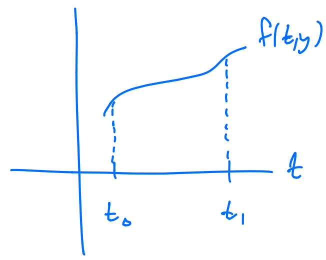
$$\epsilon_k \le \frac{Dh}{2L}\left(e^{\alpha L} - 1\right) \qquad (\S 1.13, \text{ page } 102)$$

α from E&U Thm.

$\Rightarrow$ Euler's scheme is First order Convergent. I.e. if $h \to h/2$, then $\epsilon_k \to \epsilon_k/2$.

Interpreted in terms of the exact solution:

$$y'(t) = f(t,y), \quad y(t_0) = y_0 \quad \Rightarrow \quad y(t) = y_0 + \int_{t_0}^{t} f(s,y)\,ds$$



$$\approx y_0 + (t-t_0)\cdot f(t_0, y(t_0))$$

$$= \underbrace{y_0 + h f(t_0, y_0)}.$$

Euler's Method obtained by approximating this integral

In this case, the value of f at t_0 was used. Alternatively we could have used the value at t:

$$y(t_1) = y_0 + \int_{t_0}^{t_1} f(s, y(s))\,ds$$

$$\approx y_0 + (t_1 - t_0)\, f(t_1, y(t_1))$$

Now, the equation $y_1 = y_0 + h f(t_1, y_1)$ must be solved for the value of y_1. This is known as IMPLICIT EULER.

The error is similar, but the stability is better.

Ex Stability of Euler.

Examine the model problem $y' = -\lambda y$, with $\lambda > 0$

Explicit Euler: $y_{\ell+1} = y_\ell - h\lambda y_\ell$

$$= (1 - h\lambda) y_\ell$$

The true solution is $y = ce^{-\lambda t}$, and $y(t) \to 0$ as $t \to \infty$.

In order for $y_\ell \to 0$, we require

$$y_1 = (1-h\lambda) y_0$$

$$y_2 = (1-h\lambda) y_1 = (1-h\lambda)^2 y_0$$

that $|1-h\lambda| < 1$, and therefore since $\lambda > 0$, $h > 0$,

we require $h \in (0, 2/\lambda)$ $\leftarrow$ the step size h must be in this interval to ensure stability

Implications? Example?

Implicit Euler:

$$y_{\ell+1} = y_\ell - h\lambda y_{\ell+1}$$

Solve for $y_{\ell+1}$ to obtain $y_{\ell+1} = \frac{1}{1+h\lambda} y_\ell = \frac{1}{(1+h\lambda)^{\ell+1}} y_0$

The factor $\frac{1}{1+h\lambda}$ is always < 1 if $h, \lambda > 0$, and therefore

IMPLICIT EULER is A-stable.