Oct 7, 2019

Last time:

Turn our attention to 2^{nd} order linear ODES:

$$u'' + p(x)u' + q(x)u = g(x)$$

$$\Rightarrow \quad \mathcal{L}u = g$$

$\mathcal{L}$ is a linear differential operator:

$$\mathcal{L}(c_1u_1 + c_2u_2) = c_1\mathcal{L}u_1 + c_2\mathcal{L}u_2$$

Existence & Uniqueness

Thm If p, q are continuous in $\alpha < t < \beta$, then the equation

$$(*) \qquad y'' + p(t)y' + q(t)y = 0$$
$$y(t_0) = y_0 \qquad t_0 \in (\alpha, \beta)$$
$$y'(t_0) = y_0'$$

has exactly one solution on (α, β). In particular, if $y_0 = y_0' = 0$, then $y = 0$ on (α, β).

We need several tools to analyze these equations.

Example (from last time)

$$u'' + \omega^2 u = 0 \qquad (**)$$

identity operator

$u_1 = \cos\omega x$ Since $\mathcal{L} = \frac{d^2}{dx^2} + \omega^2 I$ is linear,

$u_2 = \sin\omega x$ $u = c_1u_1 + c_2u_2$ is also a solution.

The obvious question is: are all solutions to $(**)$ of the form $c_1u_1 + c_2u_2$? Yes!

and of $(*)$, in general.

Thm: Let y_1, y_2 be solutions to $\mathcal{L}y = 0$ on (α, β). If $y_1 y_2' - y_1' y_2 \neq 0$ anywhere in (α,β), then $y = c_1 y_1 + c_2 y_2$ is the general solution to $\mathcal{L}y = 0$.

Proof: Let y be any solution to $Ly = 0$. Compute $y_0 = y(t_0)$, $y_0' = y'(t_0)$. Then we must solve for c_1, c_2 via the system:

$$y_0 = y(t_0) = c_1 y_1(t_0) + c_2 y_2(t_0)$$

$$y_0' = y'(t_0) = c_1 y_1'(t_0) + c_2 y_2'(t_0)$$

$$\Rightarrow \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

Solution exists and is unique if the determinat is non-zero:

$$y_1(t_0)\, y_2'(t_0) - y_2(t_0)\, y_1'(t_0) \neq 0.$$

Therefore, for any $t_0 \in (\alpha, \beta)$, we can compute unique c_1, c_2. ∎

Definition: The quantity $y_1 y_2' - y_1' y_2$ is called the Wronskian of y_1, y_2, denoted by $W(y_1, y_2)$.

Thm: Let p, q be continuous on (α, β) and let y_1, y_2 be two solutions to $\mathcal{L}y = 0$. Then $W(y_1, y_2) = 0$ identically, or is never equal to 0 on (α, β).

Proof: First, note that W satisfies the ODE:

$$W' + p(t)\, W = 0.$$

Just compute W':

$$W(t) = y_1(t)\,y_2'(t) - y_1'(t)\,y_2(t)$$

$$\Rightarrow \quad W' = \cancel{y_1' y_2'} + y_1 y_2'' - y_1'' y_2 - \cancel{y_1' y_2'}$$

$$= y_1 y_2'' - y_1'' y_2$$

Also, $y_1'' = -p(t)\,y_1' - q(t)\,y_1$

$y_2'' = -p(t)\,y_2' - q(t)\,y_2$

And therefore:

$$W' = y_1(-p y_2' - q y_2) - (-p y_1' - q y_1)\,y_2$$

$$= -p y_1 y_2' - \cancel{q y_1 y_2} + p y_1' y_2 + \cancel{q y_1 y_2}$$

$$= -p(y_1 y_2' - y_1' y_2)$$

$$= -p\,W$$

Given that $W' + pW = 0$, then

$$W(t) = C\,\underbrace{e^{-\int p(t)\,dt}}_{\text{never equal to zero.}}$$

therefore, either $C = 0$ and $W = 0$ everywhere, or $W \neq 0$ for any t. ▣

Definition: Two functions f, g are linearly dependent on an interval $[a,b]$ if $f(t) = c\,g(t)$ for all $t \in [a,b]$. Otherwise they are linearly independent

Thm Two solutions y_1, y_2 to $(*)$ on $[a,b]$ are linearly independent if and only if $W[y_1, y_2] \neq 0$ on $[a,b]$. Therefore, y_1 & y_2 form a fundamental solution set iff they are linearly independent.

Proof: If $W[y_1, y_2] = 0$ (everywhere), then there exists a non-zero solution to:

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\Rightarrow \quad \underbrace{c_1 y_1 + c_2 y_2}_{= y} = 0$$

$= y$, also a solution of $\mathcal{L}y = 0$, $y(t_0) = y'(t_0) = 0$,

therefore $y_1 = \frac{c_2}{c_1} y_2$.

Linear equations with constant coefficients

First special case:

$$\mathcal{L}u = a u'' + b u' + c u = 0$$

a, b, c are constants.

Ansatz: $y = C e^{rt}$, for "some" r.

Intuition: u, u', u'' must all cancel each other out when linearly combined, and therefore are likely to have "the same dependence on t"

Ex: If $u = t^n$

then $u' = n t^{n-1}$, $u'' = n(n-1) t^{n-2}$.

It is unlikely that we can choose n such that

$$a n(n-1) t^{n-2} + b n t^{n-1} + c t^n = 0.$$

However, if $u = e^{rt}$, then $u' = r e^{rt}$, $u'' = r^2 e^{rt}$ and we can find r such that $(ar^2 + br + c)\, \underset{\substack{\uparrow \\ \neq 0 \text{ ever.}}}{e^{rt}} = 0$

The equation $ar^2 + br + c = 0$ is the <u>characteristic equation</u> for $Ly = au'' + bu' + cu = 0$.

Its roots can be determined from the quadratic equation:

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$$

$$r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

If $r_1 \neq r_2$, then we have two linearly-independent solutions $u_1 = e^{r_1 t}$ and $u_2 = e^{r_2 t}$ since

$$\begin{aligned} W[u_1, u_2] &= u_1 u_2' - u_1' u_2 \\ &= e^{r_1 t} r_2 e^{r_2 t} - r_1 e^{r_1 t} e^{r_2 t} \\ &= (r_2 - r_1) e^{(r_1+r_2)t} \neq 0 \quad \text{if } r_1 \neq r_2. \end{aligned}$$

(address $r_1 = r_2$ later)

<u>Example</u>:

$$u'' - 3u' + u = 0$$

$$u'' - 3u' - 4u = 0, \quad u(0) = 1, \quad u'(0) = 0.$$