Oct 9, 2019

Last time

2^{nd} order linear differential equations:

$$u'' + p(x)u' + q(x)u = 0 \quad \text{(homogeneous case for now)}$$

- Existence & Uniqueness:
  - wherever p, q are continuous (with $u(x_0) = u_0$, $u'(x_0) = u_0'$)

- If u_1, u_2 are solutions to $\mathcal{L}u = 0$, then the general solution is $u = c_1 u_1 + c_2 u_2$ <u>if and only if</u> u_1, u_2 are linearly independent

$\Rightarrow$ check Wronskian:

$$W(u_1, u_2) = u_1 u_2' - u_1' u_2 \neq 0 \text{ if linearly independent.}$$

Also, two funcs. are linearly <u>dependent</u> on $[a,b]$ if $u_1 = c\,u_2$.

<u>Constant coefficient case</u>: $au'' + bu' + cu = 0$

Guess: $u = e^{rt}$

$\Rightarrow$ u is a solution <u>iff</u> $(ar^2 + br + c)e^{rt} = 0$

$$\Leftrightarrow \left.\underbrace{ar^2 + br + c = 0}_{\text{characteristic equation}}\right\} \text{ Solve using quadratic formula}$$

- If $r_1 \neq r_2$, then $W(e^{r_1 t}, e^{r_2 t}) \neq 0$.

- Two remaining cases: (1) r_1, r_2 are complex

  (2) $r_1 = r_2$

Complex-valued Roots

Often times, we will compute r_1, r_2 that are complex.

This happens when $4ac > b^2$.

In this case, the roots are always of the form

$$r_1 = \alpha + i\beta \qquad r_2 = \alpha - i\beta.$$

And therefore

$$u_1 = e^{r_1 t} \qquad u_2 = e^{r_2 t}$$

$$= e^{(\alpha + i\beta)t} \qquad = e^{(\alpha - i\beta)t}$$

$$= \overline{u_2} \qquad = \overline{u_1} \quad \text{(complex conjugate)}$$

But what if we want a real solution?

Lemma: If $u(t) = x(t) + i\,y(t)$, with x, y real-valued, and $\mathcal{L}u = 0$, then x, y are both real-valued solutions.

Proof. Since $\mathcal{L}$ is a linear operator.

Taking real/imag parts of u_1, u_2:

$$\left.\begin{aligned} u_1 &= e^{\alpha t}\left(\cos\beta t + i\sin\beta t\right) \\ u_2 &= e^{\alpha t}\left(\cos\beta t - i\sin\beta t\right) \end{aligned}\right\}$$

Prove this formula using Taylor series...

$$e^{i\beta t} = 1 + (i\beta t) + \frac{(i\beta t)^2}{2} + \ldots$$

Therefore, two real-valued linearly independent solutions are:

$$y_1 = e^{\alpha t}\cos\beta t \quad \text{and} \quad y_2 = e^{\alpha t}\sin\beta t.$$

Check: $W[y_1, y_2] = \ldots$ (exercise for the reader).

Example: $u'' + 2u' + 3u = 0$

$u'' + u' + 2u = 0, \quad u(0) = 1, \quad u'(0) = -2$

Repeated Roots

What if $r_1 = r_2$? $\qquad$ Ex: $u'' - 2u' + u = 0$

$$p(r) = r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0 \implies r_1 = r_2 = 1.$$

Idea: Given that $u_1 = u(t)$ solves $\mathcal{L}u = 0$, can we find another solution?

Let $u_2(t) = u(t)v(t)$

Compute derivatives:

$$u_2' = uv' + u'v$$

$$u_2'' = uv'' + u'v' + u'v' + u''v$$

$$= uv'' + 2u'v' + u''v$$

The function u_2 is a solution to $\mathcal{L}u = 0 = u'' + pu' + qu$ if

$\mathcal{L}u_2 = 0$

$$= uv'' + 2u'v' + u''v + p(uv' + u'v) + quv$$

$$= uv'' + (2u' + pu)v' + \underbrace{(u'' + pu' + qu)}_{\mathcal{L}u = 0}v$$

$$= uv'' + (2u' + pu)v'$$

This is just a first order equation in terms of the function v':

$$u(v')' + (2u' + pu)(v') = 0$$

The solution can be obtained by separation:

$$\frac{(v')'}{(v')} = \frac{-(2u' + pu)}{u}$$

$$\Rightarrow \quad v' = C\, e^{-\int 2\frac{u'}{u}}\, e^{-\int p}$$

$$2\int \frac{u'(t)}{u(t)}\,dt = 2\int \frac{d}{dt}(\log|u|)\,dt$$
$$= 2\log|u| = \log|u|^2$$

$$= C\, e^{-\log u^2}\, e^{-\int p}$$

$$= \frac{C\, e^{-\int p}}{u^2}$$

We only need <u>one</u> v, so we set $C=1$ and then integrate:

$$v' = \frac{1}{u^2}\, e^{-\int p}$$

$$\Rightarrow \int v' = \int \frac{1}{u^2}\, e^{-\int p}$$

$$\Rightarrow \quad v = \underbrace{\int \frac{1}{u^2(t)}\, e^{\overbrace{-\int p(t)\,dt}^{\text{anti-derivative}}}\,dt}_{\text{anti-derivative.}}$$

This method is called <u>The Method of Reduction of Order</u> since the change of variable $u_2 = uv$ required only the solution to a <u>1st</u> order equation to obtain v.

Application: Equal roots in the characteristic equation for $a u'' + b u' + c u = 0$.

In this case, rewrite as:

$$u'' + \underbrace{\frac{b}{a}}_{p} u' + \underbrace{\frac{c}{a}}_{q} u = 0$$

Then $u_1 = e^{rt}$, and set $u_2 = u_1 v$.

By the above calculation,

$$v = \int \frac{1}{u^2} e^{-\int p\,dt}\,dt$$

$$= \int \frac{1}{e^{2rt}} e^{-\int \frac{b}{a}\,dt}\,dt = \int e^{-2rt} e^{-\frac{b}{a}t}\,dt$$

But if the root was repeated, then $b^2 = 4ac$, and $r = \frac{-b}{2a}$,

so

$$v = \int e^{+2\frac{b}{2a}t} e^{-\frac{b}{a}t}\,dt$$

$$= \int 1\,dt = t$$

Therefore, a second linearly independent solution is $u_2 = t e^{rt}$.

Check:

$$u_2' = e^{rt} + rt e^{rt}$$

$$u_2'' = re^{rt} + re^{rt} + r^2 t e^{rt}$$

Recall: $r = \frac{-b}{2a}$

so

$$\mathcal{L}u_2 = \left(a\left(-\frac{b}{a} + \frac{b^2}{4a^2}t\right) + b\left(1 - \frac{b}{2a}t\right) + ct \right)$$

but $c = \frac{b^2}{4a}$

$$= -b + \frac{b^2}{4a}t + b - \frac{b^2}{2a}t + \frac{b^2}{4a}$$

$$= 0.$$

<u>Application</u> Other differential equations: (non-constant coefficients)

Ex: $(1-t^2)y'' + 2ty' - 2y = 0$

Verify: $y_1 = t$ is one solution.

Rewrite to apply the Method of Reduction of Order:

$$y'' + \underbrace{\frac{2t}{1-t^2}}_{p} y' - \underbrace{\frac{2}{1-t^2}}_{q} y = 0.$$

Other solution is $y_2 = y_1 v = tv$

$$v = \int \frac{1}{t^2} e^{-\int \frac{2t}{1-t^2}dt} dt$$

$$= \int \frac{1}{t^2} e^{\log(1-t^2)} dt$$

$$= \int \frac{1-t^2}{t^2} dt = -\frac{1}{t} - t$$

$$\Rightarrow \quad y_2 = -t\left(\frac{1}{t} + t\right) = -(1+t^2)$$