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Last time:

Constant coefficient 2^{nd}-order Lin. DE:

$$au'' + bu' + cu = 0$$

$\Rightarrow$ $u = e^{rt}$, $p(r) = ar^2 + br + c = 0$ Characteristic Equation.

① Two real, distinct roots

$\Rightarrow$ $u_1 = e^{r_1 t}$, $u_2 = e^{r_2 t}$

② Two complex roots , $r = \alpha \pm i\beta$

$\Rightarrow$ $u_1 = e^{\alpha t} \cos \beta t$, $u_2 = e^{\alpha t} \sin \beta t$

③ Repeated root (real)

$\Rightarrow$ $u_1 = e^{rt}$, $u_2 = t e^{rt}$

③ Was derived by assuming $u_2 = u_1(t) \cdot v(t)$, solved for $v(t)$. <u>Method of Reduction of Order</u>.

$$v(t) = \int \frac{1}{u_1(t)^2} e^{-\int p(t)\,dt} \, dt$$

Applicable to general equations of the form:

$$u'' + p(x) u' + q(x) u = 0.$$

<u>Next topic</u>

The inhomogeneous equation:

$$u'' + p(x) u' + q(x) u = g(x).$$

The solution will be composed of two pieces:

$$u = \underbrace{u_h}_{\text{solves } g=0} + \underbrace{u_p}_{\substack{\text{solves } g \neq 0 \\ \text{"Particular solution"}}}$$

Ex: $u' - 2xu = x$

Sol to $u' - 2xu = 0$ is $u_h(x) = e^{x^2}$

A Sol to $u' - 2xu = x$ is $u_p(x) = -\frac{1}{2}$ ← particular solution.

$$\Rightarrow \quad u(x) = ce^{x^2} - \frac{1}{2}.$$

So if u_1, u_2 are the fundamental solution set to

$u'' + p(x)u' + q(x)u = 0$, then any solution to

(*) $u'' + p(x)u' + q(x)u = g(x)$ must be of the form:

$$u(x) = c_1 u_1(x) + c_2 u_2(x) + \underbrace{\psi(x)}_{\text{particular solution.}}$$

Lemma If u, v solve (*), then $w = u - v$ must solve the homogeneous equation $\mathcal{L}u = 0$:

Proof: $\mathcal{L}w = \mathcal{L}(u-v) = \mathcal{L}u - \mathcal{L}v = g - g = 0.$

Example: $u'' + u = 2x$

Homogeneous solutions are $u_1 = \cos x$, $u_2 = \sin x$

And clearly $u_p = 2x$.

$$\Rightarrow \quad u(x) = c_1 \cos x + c_2 \sin x + 2x.$$

This is great, but how do we find the particular solutions?

Method of Variation of Parameters

Let the solutions to $\mathcal{L}y = 0$ be y_1, y_2. textbook notation

Ansatz: $y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x) = \psi(x)$

unknown functions.

"the varying parameters"

Idea: With y_1, y_2 known, the equation $\mathcal{L}y = g$ is only one equation for the two unknowns u_1, u_2. Let us choose the other condition so that things simplify:

$$\psi = u_1 y_1 + u_2 y_2$$

$$\psi' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

$$= (u_1 y_1' + u_2 y_2') + (u_1' y_1 + u_2' y_2)$$

If $u_1' y_1 + u_2' y_2 = 0$, then ψ'' contains no 2nd-order derivatives of u_1, u_2. This is one additional constraint (equation) used to determine u_1, u_2.

Compute $\mathcal{L}\psi$ using this constraint:

$$\mathcal{L}\psi = \psi'' + p(x)\psi' + q(x)\psi$$

$$= u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'' + p(x)(u_1 y_1' + u_2 y_2')$$

$$+ q(x)(u_1 y_1 + u_2 y_2)$$

$$= u_1(y_1'' + p y_1' + q y_1) + u_2(y_2'' + p y_2' + q y_2)$$

$$+ u_1' y_1' + u_2' y_2'$$

$$= u_1 \cancel{\mathcal{L}y_1} + u_2 \cancel{\mathcal{L}y_2} + u_1' y_1' + u_2' y_2'$$

$$= u_1' y_1' + u_2' y_2'$$

So: u_1, u_2 must satisfy:

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = g \qquad \text{in order for } \mathcal{L}\psi = g.$$

$$\Rightarrow \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

The solution can be obtained by Gaussian Elimination:

$$u_1' = \frac{-g\,y_2}{y_1y_2' - y_1'y_2} \qquad u_2' = \frac{g\,y_1}{y_1y_2' - y_1'y_2}$$

$$= \frac{-g\,y_2}{W(y_1,y_2)} \qquad = \frac{g\,y_1}{W(y_1,y_2)}$$

Wronskian

And u_1, u_2 can be obtained by integrating these expressions.

Example:

$$\underbrace{2y'' - 3y' + y}_{\substack{\text{constant coefficients} \\ \mathcal{L}y}} = (x^2+1)e^x$$

$\Rightarrow$ Solve using characteristic equation:

$$2r^2 - 3r + 1 = 0$$

$$r = \frac{3 \pm \sqrt{9 - 4\cdot 2\cdot 1}}{4} = \frac{3 \pm 1}{4} = 1,\ \tfrac{1}{2}$$

$\Rightarrow$ $y_1(x) = e^x$, $y_2(x) = e^{x/2}$

By the previous variation of parameters calculation, we have that

$$W(y_1,y_2) = y_1y_2' - y_1'y_2$$

$$= \left(\tfrac{1}{2} - 1\right) e^x e^{x/2} = -\tfrac{1}{2} e^{3x/2}$$

$$g = (x^2+1)e^x$$

$$u_1' = -\frac{(x^2+1)e^x e^{x/2}}{-\frac{1}{2}e^{3x/2}} = 2(x^2+1)$$

$$\Rightarrow u_1 = \tfrac{2}{3}x^3 + 2x$$

$$u_2' = \frac{(x^2+1)e^x e^x}{-\frac{1}{2}e^{3x/2}} = -2(x^2+1)e^{x/2}$$

$$\Rightarrow u_2 = -\int 2(x^2+1)e^{x/2}\,dx$$

$$= -2\left(2x^2 - 8x + 18\right)e^{x/2}$$

<u>Example:</u> $y'' + y = \sec x$

Homogeneous solution: $(r^2+1) = 0$

$$\Rightarrow y_1 = \cos x$$
$$y_2 = \sin x$$

Look for particular solution of the form $\psi = u_1 y_1 + u_2 y_2$.

By the previous calculation

$$u_1' = \frac{g y_2}{y_1 y_2' - y_1' y_2} \qquad u_2' = \frac{g y_1}{y_1 y_2' - y_1' y_2}$$

$$y_1 y_2' - y_1' y_2 = \cos^2 x + \sin^2 x = 1$$

$$u_1' = \sec x \cdot \sin x$$

$$= \frac{\sin x}{\cos x} = \tan x \quad \Rightarrow \quad u_1 = \int \tan x\,dx = \log \sec x$$

$$u_2' = \sec x \cdot \cos x = 1 \quad \Rightarrow \quad u_2 = \int dx = x$$

$$\Rightarrow y = \underbrace{c_1 \cos x + c_2 \sin x}_{\text{homogeneous soln}} + \underbrace{(\log \sec x)\cos x + x \sin x}_{\text{particular solution}}$$