Oct 23, 2019

Last time

Singular Equations.

Euler's Equation:

$$t^2 y'' + \alpha t y' + \beta y = 0$$

Ansatz: $y(t) = t^r$, plug in and solve for r.

3 cases: (1) two real, distinct roots

(2) two complex roots (take real, imag. parts)

(3) repeated roots (use method of reduction of order)

For $t<0$, use change of variable to show that solutions are a function of $|t|$.

Then Generalize Euler's equation by replacing α, β with polynomials:

$$t^2 y'' + t\left(\sum_0^\infty p_n t^n\right) y' + \left(\sum_0^\infty q_n t^n\right) y = 0$$

$$\Rightarrow \quad y'' + \underbrace{\left(\sum_0^\infty p_n t^{n-1}\right)}_{p(x)} y' + \underbrace{\left(\sum_0^\infty q_n t^{n-2}\right)}_{q(x)} y = 0$$

When p, q are of this form, $t=0$ is referred to as a Regular Singular Point.

Frobenius Method: Use the ansatz:

$$y(t) = \underset{\substack{\uparrow \\ \text{from Euler}}}{t^r} \underbrace{\sum_0^\infty a_n t^n}_{\text{from general series solution}}$$

Best shown using an example.

Example: $2tu'' + u' + tu = 0$

Assume $u = \sum_0^{\infty} a_n t^{r+n}$ $\Rightarrow$ $u' = \sum_0^{\infty} (r+n) a_n t^{r+n-1}$

$u'' = \sum_0^{\infty} (r+n)(r+n-1) a_n t^{r+n-2}$

$$\Rightarrow \sum_0^{\infty} 2(r+n)(r+n-1) a_n t^{r+n-1} + \sum_0^{\infty} (r+n) a_n t^{r+n-1} + \sum_0^{\infty} a_n t^{r+n+1} = 0$$

$$\Rightarrow t^r \Big(2r(r-1)a_0 t^{-1} + 2(r+1)(r)a_1 + \sum_2^{\infty} 2(r+n)(r+n-1) a_n t^{n-1} + r a_0 t^{-1} + (r+1)a_1$$
$$+ \sum_2^{\infty} (r+n) a_n t^{n-1} + \sum_2^{\infty} a_{n-2} t^{n-1} \Big) = 0$$

Now collect terms, set to 0:

① $\left(2r(r-1)a_0 + r a_0\right) t^{-1} = 0$ $\Rightarrow$ $\left(r(2r-1)a_0\right) = 0$

$\Rightarrow$ $r = 0$ or $r = \frac{1}{2}$

② $\left(2(r+1)r\, a_1 + (r+1) a_1\right) = 0$ $\Rightarrow$ $r = 0 \Rightarrow a_1 = 0$

$r = \frac{1}{2} \Rightarrow a_1 = 0$

③ $\left(2(r+n)(r+n-1) a_n + (r+n) a_n + a_{n-2}\right) t^{n-1} = 0$ for $n = 2, 3, \ldots$

$(r+n)\left(2(r+n-1) + 1\right) a_n = -a_{n-2}$

$$\Rightarrow \quad a_n = \frac{-a_{n-2}}{(r+n)(2(r+n-1)+1)}$$

Two recurrence relationships, one for $r=0$ and one for $r = \frac{1}{2}$.

$$r = 0 \Rightarrow a_n = \frac{-a_{n-2}}{n(2n-1)}$$

$$r = \tfrac{1}{2} \Rightarrow a_n = \frac{-a_{n-2}}{(n+\frac{1}{2})(2(n-\frac{1}{2})+1)} = \frac{-a_{n-2}}{(2n+1)n}$$

Note Only one free coefficient : a_0

two options for r : $r = 0, \frac{1}{2}$

two solutions :

$$x(t) = \sum_{n=0}^{\infty} a_n t^n + \sqrt{t} \sum b_n t^n \quad \leftarrow \text{General solution.}$$

$a_{2n+1} = 0$ $b_{2n+1} = 0$

$a_n = \dfrac{-a_{n-2}}{n(2n-1)}$ $b_n = \dfrac{-b_{n-2}}{(2n+1)n}$

a_0 free b_0 free.

This is the method of Frobenius (Ansatz $x(t) = \sum a_n t^{n+r}$, for $x'' + px' + qx = 0$, regular singular point at $t=0$).

What about the general case in which

$$p(t) = \frac{p_0}{t} + p_1 + p_2 t + p_3 t^2 + \ldots$$

$$q(t) = \frac{q_0}{t^2} + \frac{q_1}{t} + q_2 + q_3 t + q_4 t^2 + \ldots$$

Inserting the Frobenius ansatz into the equation we see that:

$$\sum_0^{\infty} (r+n)(r+n-1) a_n t^{r+n-2} + \left(\sum_0^{\infty} p_n t^{n-1} \right) \left(\sum_0 (r+n) a_n t^{r+n-1} \right)$$
$$+ \left(\sum_0^{\infty} q_n t^{n-2} \right) \left(\sum_0 a_n t^{n+r} \right) = 0$$

Multiply through by t^2:

$$\sum_0^{\infty} \left((r+n)(r+n-1) a_n + \left(\sum_{m=0}^{\infty} p_m t^m \right) (r+n) a_n + \left(\sum_{m=0}^{\infty} q_m t^m \right) a_n \right) t^{n+r} = 0$$

The equation that will determine the values of r is the coefficient of the t^r term in the above expression (i.e. $n=0$):

This is given by:

$$\underbrace{\left(r(r-1) + p_0 r + q_0\right)}a_0 = 0$$

Quadratic equation for r, denoted by $F(r)$ and known as the Indicial Equation.

Examine the t^{r+1} equation:

$$(r+1)r\,a_1 + p_0(r+1)a_1 + p_1 r a_0 + q_0 a_1 + q_1 a_0 = 0$$

$$\Rightarrow \quad \underline{F(r+1)}\,a_1 = -p_1 r a_0 - q_1 a_0$$

In general each equation can be written as:

$$F(n+r)\,a_n = -\sum_{k=0}^{n-1}\left((k+r)p_{n-k} + q_{n-k}\right)a_k$$

This means that each coefficient depends on all previous coefficients. Closed-form solutions happen very rarely.

We will not discuss the case in which $F(r)=0$ has complex or repeated roots, as details are straightforward but tedious.

Next topic: Laplace Transforms.