Oct 28, 2019

Last time:

Method of Frobenius: (Generalizes solutions to Euler's eq'n)

For an equation of the form

$t=0$ is a regular singular point

$$t^2 u'' + t\,p(t)\,u' + q(t)\,u = 0$$

with $p = \sum_0^\infty p_n t^n$, $q = \sum_0^\infty q_n t^n$ search for a solution of the form:

possibly fractional powers

$$u(t) = \sum_{n=0}^\infty a_n t^{r+n}$$

Substitute in, solve for r and the a_n's via resulting recurrence formula. The power r is determined by the Indicial Equation:

$$F(r) = r(r-1) + p_0 r + q_0 = 0$$

The two solutions are determined by the two roots of F.

Usually a_0 serves as the unknown constant in the solution.

Mechanically, very similar to solving Euler's Equation or finding a series solution to a non-singular equation.

Next topic Laplace Transforms

Motivating example:

When solving $A\vec{x} = \vec{b}$ ($n\times n$ linear system), if we knew A^{-1}, then just apply it to both sides:

$$A^{-1}A\vec{x} = \vec{x} = A^{-1}\vec{b}. \quad \text{Done}$$

Alternatively, if we knew some other matrix which had

B

the property that

$$BA = \text{diagonal}$$
or
$$BA = \text{tridiagonal}$$

each of these make the linear system very easy to solve:

$$(BA)\vec{x} = B\vec{b}.$$

Can something similar be done for DE? Can we find some other linear operator $\mathcal{T}$ such that

$$\mathcal{T}\mathcal{L}u = \mathcal{T}g$$ is <u>much</u> easier to solve than $\mathcal{L}u = g$?

<u>Definition</u>: The Laplace Transform of a function f defined on $0 \le t < \infty$ is

$$F(s) = \mathcal{T}f(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \lim_{A\to\infty} \int_0^A e^{-st} f(t)\, dt$$

<u>Example</u>: $f = 1$

$$F(s) = \int_0^\infty e^{-st}\, dt = -\frac{1}{s} e^{-st}\Big|_0^\infty = \begin{cases} 1/s & \text{if } s > 0 \\ \infty & \text{if } s \le 0 \end{cases}$$

<u>Example</u>: $f(t) = e^{i\omega t}$, ω real

$$F(s) = \int_0^\infty e^{-st} e^{i\omega t}\, dt = \int_0^\infty e^{(i\omega - s)t}\, dt$$

$$= \frac{1}{i\omega - s} e^{(i\omega - s)t}\Big|_0^\infty = \begin{cases} -\frac{1}{i\omega - s} & \text{if } s > 0 \\ \infty & \text{if } s \le \infty \end{cases}$$

$$\Rightarrow \frac{-1}{i\omega - s} = \frac{1}{s - i\omega} \frac{s + i\omega}{s + i\omega} = \frac{s + i\omega}{s^2 + \omega^2} = \frac{s}{s^2+\omega^2} + i\frac{\omega}{s^2+\omega^2}$$

Since $e^{i\omega t} = \cos\omega t + i\sin\omega t$, we have

$$\mathcal{T}(\cos\omega t)(s) = \frac{s}{s^2+\omega^2}, \quad \mathcal{T}(\sin\omega t)(s) = \frac{\omega}{s^2+\omega^2}.$$

Assumptions on f in order for $\mathcal{T}f$ to exist:

Lemma If f is piecewise continuous and there exists $M, C > 0$ such that $|f(t)| < M e^{ct}$, $t \geq 0$, then $\mathcal{T}f(s)$ exists for $s > c$.

Sketch: $F(s) = \int_0^\infty e^{-st} f(t)\, dt \leq \int_0^\infty e^{-st} |f(t)|\, dt$

$$< \int_0^\infty e^{-st} M e^{ct}\, dt$$

$$= M \int_0^\infty e^{(c-s)t}\, dt < \infty \text{ iff } s > c.$$

We will apply the operator $\mathcal{T}$ to both sides of $\mathcal{L}u = g$. In order to do this, we need the quantity: $\mathcal{T}u' = \mathcal{T}\frac{du}{dt}$.

$\mathcal{T}u' = \int_0^\infty e^{-st} u'(t)\, dt$, integrate by parts:

$$\Rightarrow \quad = e^{-st} u(t)\Big|_0^\infty - \int_0^\infty \frac{d}{dt}\left(e^{-st}\right) u(t)\, dt$$

$$= -u(0) + s\int_0^\infty e^{-st} u(t)\, dt = -u(0) + s\,\mathcal{T}u(s).$$

Therefore, if $F(s) = (\mathcal{T}f)(s)$, then $(\mathcal{T}f')(s) = -f(0) + sF(s)$.

Likewise: $(\mathcal{T}f'')(s) = s^2 F(s) - s f(0) - f'(0)$.

(Proof also by integrating by parts.)

Aside: To remember integration by parts:

$$(uv)' = u'v + uv'$$

Integrate both sides:

$$\int_a^b (uv)' = \int_a^b u'v + \int_a^b uv'$$

$$\Rightarrow \int_a^b uv' = uv\Big|_a^b - \int_a^b u'v$$

Now apply to 2^{nd} order constant coefficient inhomogeneous DE:

$$\mathcal{T}(au'' + bu' + cu) = \mathcal{T}g$$

$$\Rightarrow a\left(s^2U(s) - su(0) - u'(0)\right) + b\left(sU(s) - u(0)\right) + cU(s) = G(s).$$

$\Rightarrow$ solve for $U(s)$:

$$\left(as^2 + bs + c\right)U(s) = G(s) + a\left(su(0) + u'(0)\right) + bu(0)$$

$$\Rightarrow U(s) = \underbrace{\frac{G(s) + a\left(su(0) + u'(0)\right) + bu(0)}{\left(as^2 + bs + c\right)}}$$

If this quantity is the known Laplace transform of some function $\phi(t)$, then the solution must be $u(t) = \phi(t)$.

Under suitable conditions, the function $u(t)$ can be determined directly from $U(s)$ by applying the Inverse Laplace Transform:

$$u(t) = \left(\mathcal{T}^{-1}U\right)(t).$$

The computation of $\mathcal{T}^{-1}U$ requires complex analysis and is therefore beyond our scope. We must use Tables of Known Laplace Transform Pairs.

$$\left(\mathcal{T}^{-1}U(t) = \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st}\, U(s)\, ds \quad , \quad \text{Fourier-Mellin integral} \right).$$

Example: $y'' - 5y' + 4y = e^{2t}$, $y(0) = 1$, $y'(0) = -1$.

Taking Laplace Transform of both sides:

$$s^2Y(s) - sy(0) - y'(0) - 5\left(sY(s) - y(0)\right) + 4Y(s) = \frac{1}{s-2}, \quad \text{for } s > 2.$$

$$s^2Y(s) - s + 1 - 5sY(s) + 5 + 4Y(s) = \frac{1}{s-2}$$

$$Y(s)\left(s^2 - 5s + 4\right) = \frac{1}{s-2} + s - 6$$

$$Y(s)(s-4)(s-1) = \frac{1}{s-2} + s - 6$$

$$Y(s) = \frac{1}{(s-2)(s-4)(s-1)} + \frac{s}{(s-4)(s-1)} - \frac{6}{(s-4)(s-1)}$$

Partial fractions:

$$\frac{6}{(s-4)(s-1)} = \frac{+2}{(s-4)} + \frac{-2}{(s-1)} \qquad \left(\frac{2(s-1) - 2(s-4)}{(s-4)(s-1)} = \frac{2s-2-2s+8}{(s-4)(s-1)} = \frac{6}{\sim} \right)$$

$$\frac{s}{(s-4)(s-1)} = \frac{1}{3}\left(\frac{4}{s-4} - \frac{1}{s-1}\right)$$

$a(s-1) + b(s-4) = s + 0$

$a + b = 1 \quad \rightarrow -3b = 1 \qquad b = -\frac{1}{3}$

$-a - 4b = 0 \qquad a = -4b \qquad a = \frac{4}{3}$

$$\frac{1}{(s-2)(s-4)(s-1)} = -\frac{1}{2}\frac{1}{s-2} + \frac{1}{3}\frac{1}{s-1} + \frac{1}{6}\frac{1}{s-4}$$

Therefore, $Y(s) = -\frac{1}{2}\frac{1}{s-2} + \left(2 - \frac{1}{3} + \frac{1}{3}\right)\frac{1}{s-1} + \left(-2 + \frac{4}{3} + \frac{1}{6}\right)\frac{1}{s-4}$

$$= -\frac{1}{2}\frac{1}{s-2} + 2\frac{1}{s-1} - \frac{1}{2}\frac{1}{s-4}$$

And so it must be that $\underline{y(t) = -\frac{1}{2}e^{2t} + 2e^{t} - \frac{1}{2}e^{4t}}$.

Properties of Laplace Transforms

Prop. 1: If $F(s) = \int_0^\infty e^{-st} f(t)\, dt$, then

$$\int_0^\infty e^{-st}\left(-t f(t)\right) dt = F'(s).$$

Straightforward computation:

$$F'(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty \left(\frac{\partial}{\partial s} e^{-st}\right) f(t)\, dt$$

$$= -\int_0^\infty t\, e^{-st} f(t)\, dt.$$

<u>Example</u> Previously, we saw that

$$\mathcal{L}(e^{at})(s) = \frac{1}{s-a} \quad , \; s > a$$

So $\mathcal{L}(t\, e^{at})(s) = -\frac{d}{ds}\frac{1}{s-a} = \frac{1}{(s-a)^2}$