Nov 13, 2019

Last time:

Systems of linear differential equations:

Concise notation: $\vec{x}' = A\vec{x} + \vec{g}$

$A_{ij} = a_{ij}(t)$ If a_{ij} is constants, system is const. coeff.

$g_j = g_j(t)$ If $\vec{g} = \vec{0}$, then system is homogeneous.

Example:

$$y'' + py' + qy = g$$

Let $x_1 = y$, $x_2 = y'$ $\Rightarrow$ $x_1' = x_2$, $x_2' = g - px_2 - qx_1$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ g \end{pmatrix}$$

① Existence and Uniqueness:

If A is constant, then solution to $\vec{x}' = A\vec{x}$, $\vec{x}(0) = \vec{x}_0$ is unique and exists for all t.

② Dimension of vector space of solutions to $\vec{x}' = A\vec{x}$ is $\underline{n}$ (i.e. n linearly independent solutions).

(We proved this.)

Thm: Let $\vec{x}_1, \dots, \vec{x}_k$, $k \le n$, be k solutions to $\vec{x}' = A\vec{x}'$.
The solutions $\vec{x}_1, \dots, \vec{x}_k$ are linearly independent <u>if and only if</u> the vectors $\vec{x}_1(t_0), \dots, \vec{x}_k(t_0) \in \mathbb{R}^n$ are linearly independent. (for any t_0).

Proof (straight forward, just evaluate $\vec{x}_1, \dots, \vec{x}_k$ at t_0).

Example:

$$y'' + 2y' + y = 0 \quad \Rightarrow \quad \text{solutions can be computed as } y_1(t) = e^{-t}, \quad y_2(t) = te^{-t}$$

Let $x_1 = y$, $x_2 = y'$

$$\Rightarrow \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} x_2 \\ -2x_2 - x_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\Rightarrow$ Solution is $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix}$

$$\Rightarrow \vec{x}^1 = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}, \quad \vec{x}^2 = \begin{pmatrix} te^{-t} \\ (1-t)e^{-t} \end{pmatrix}.$$

Check that are linearly independent.

Eigenvector / eigenvalue solution method

Recall:

If A is an $n \times n$ matrix, λ and $\vec{v}$ are an eigenvalue / eigenvector pair if:

$$A\vec{v} = \lambda\vec{v}$$

To find: $\Rightarrow$

$$A\vec{v} - \lambda\vec{v} = \vec{0}$$
$$\underbrace{(A - \lambda I)}\vec{v} = \vec{0}$$

(1) Find λ such that $(A - \lambda I)^{-1}$ does <u>not</u> exist

(2) Find nullspace of this matrix

Applying this idea to diff. eqns.:

$$y' = \lambda y \quad \Rightarrow \quad \left.\begin{array}{l} y = e^{\lambda t} \\ y' = \lambda e^{\lambda t} \end{array}\right\} \Rightarrow$$

$e^{\lambda t}$ is an eigenfunction of the differentiation operator.

Applied to a system of diff. eqns:

$\underline{\vec{x}' = A\vec{x}}$ Look for a solution of the form

$$\vec{x} = e^{\lambda t}\,\vec{v}.$$

constant vector.

Then $\vec{x}' = \lambda e^{\lambda t}\,\vec{v}$ is a solution.

If both $\lambda_1, \vec{v}_1$ and $\lambda_2, \vec{v}_2$ are eigen pairs with $\vec{v}_1, \vec{v}_2$ linearly independent, then $\vec{x} = c_1 e^{\lambda t}\vec{v}_1 + c_2 \lambda$

<u>Thm</u> Any k eigenvectors $\vec{v}_1, \ldots, \vec{v}_k$ of A with distinct eigenvalues are linearly independent.

<u>Proof</u> Done in text and linear algebra course.

If A has n linearly independent eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$, the general solution to

$\vec{x}' = A\vec{x}$ is given by

$$\vec{x} = c_1 e^{\lambda_1 t}\vec{v}_1 + \ldots + c_n e^{\lambda_n t}\vec{v}_n$$

If the initial condition $\vec{x}(0) = \vec{x}_0$ is to be satisfied, then

$$\vec{x}_0 = c_1 e^{\lambda_1 \cdot 0}\vec{v}_1 + \ldots + c_n e^{\lambda_n \cdot 0}\vec{v}_n$$

$$= \underbrace{\begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}}_{\text{Solve this linear system for } \vec{c}.} = V\vec{c}$$