December 9, 2019

New topic: Two-point boundary value problems:

Consider:

$$\underbrace{y'' + \lambda y = 0}_{\text{ODE}} \qquad \underbrace{\begin{matrix} a y(0) + b y'(0) = 0 \\ c y(1) + d y'(1) = 0 \end{matrix}}_{\text{boundary conditions (B.C.'s)}}$$

For what values of λ are there non-trivial (non-zero) solutions?

This is an eigenvalue problem:

$$y'' = -\lambda y \quad + \quad \text{B.C.'s.}$$

Very different than an initial value problem.

Ex. $y'' + \lambda y = 0$
$y(0) = 0$
$y(\ell) = 0$

Case 1 If $\lambda = 0$, then $y = c_1 x + c_2$
$\Rightarrow y = 0$ is only solution

Case 2 If $\lambda < 0$, then $y = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$

Applying BC's: $c_1 + c_2 = 0$
$c_1 e^{\sqrt{-\lambda}\ell} + c_2 e^{-\sqrt{-\lambda}\ell} = 0$

$\Rightarrow c_1 = -c_2$

So $c_1 \underbrace{\left(e^{\sqrt{-\lambda}\ell} - e^{-\sqrt{-\lambda}\ell}\right)}_{>0 \text{ for } \ell > 0} = 0 \quad \Rightarrow \quad c_1 = c_2 = 0.$

Case 2

$\lambda > 0$, then $y = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x$

$y(0) = 0 \Rightarrow c_1 = 0$

$y(\ell) = 0 \Rightarrow c_2 \sin\sqrt{\lambda}\,\ell = 0 \Rightarrow$ Either $c_2 = 0$, and $y = 0$

OR that $\sqrt{\lambda}\,\ell = n\pi$

$\Rightarrow \lambda = \dfrac{n^2\pi^2}{\ell^2}$

Therefore the 2-point BVP $\begin{array}{l} y'' + \lambda y = 0 \\ y(0) = y(\ell) = 0 \end{array}$ has

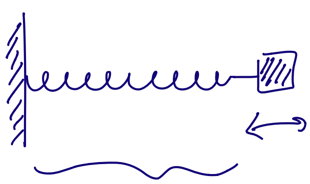
non-trivial solutions if $\lambda = \dfrac{n^2\pi^2}{\ell^2}$, $n = 1, 2, \ldots$, and in which case the solution is $y(x) = \sin \dfrac{n\pi}{\ell} x$.

The set of eigenvalues and eigenfunctions to this problem are

$$\underbrace{\frac{n^2\pi^2}{\ell^2}}_{\lambda_n}, \quad \underbrace{\sin\frac{n\pi}{\ell}x}_{\varphi_n(x)}$$

Rewriting: $-y'' = -\dfrac{d^2}{dx^2} y = \mathcal{L} y = \lambda y$

Physics application:



spring satisfies an ODE, at which forcing frequencies will there be "standing waves"?

Other applications come from PDEs...

Partial Differential Equations

Relates changes in multiple variables (time, space, etc.).

Three main PDEs:

$\Delta u = 0$	$\frac{\partial u}{\partial t} = \Delta u$	$\frac{\partial^2 u}{\partial t^2} = \Delta u$
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$	$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$	
Laplace Eqn	Heat Equation	Wave Equation

Each equation must be augmented with boundary conditions (in the spatial variables) and initial conditions (in the temporal variable).

Example Heat equation

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \quad + \quad u(x,0) = f(x) \quad \text{Initial condition}$$

$$u(0,t) = u(L,t) = 0 \quad \text{Boundary condition}$$

α^2: diffusion constant, $\alpha > 0$.

One ansatz for the solution: $u(x,t) = \underbrace{X(x)\,T(t)}_{\text{separation of variables.}}$

Inserting into the equation we have:

$$X T' = \alpha^2 X'' T$$

$$\underbrace{\frac{T'}{\alpha^2 T}}_{\text{only depends on } t} = \underbrace{\frac{X''}{X}}_{\text{only depends on } X} \Rightarrow \frac{T'}{T} \text{ and } \frac{X''}{X} \text{ must be constant}$$

$$\Rightarrow \frac{T'}{\alpha^2 T} = -\lambda = \frac{X''}{X}$$

The initial condition implies: $u(x,0) = X(x)T(0) = f(x)$.

The B.C. implies: $u(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0$

$u(L,t) = X(L)T(t) = 0 \qquad X(L) = 0$

Therefore $u(x,t) = X(x)T(t)$ only if

$X'' + \lambda X = 0$, $X(0) = 0$, $X(L) = 0$ } BVP, only has non-trivial solutions if λ is eigenvalue.

and $T' + \lambda\alpha^2 T = 0$

I.E. $\lambda_n = \frac{n^2\pi^2}{L^2}$, $X_n(x) = \sin\frac{n\pi}{L}x$

Using these values of λ_n, it also must be that $T_n(t) = e^{-\alpha^2 n^2 \pi^2 t/L^2}$

So u consists of any linear combination of $u_n = X_n T_n$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} a_n u_n(x,t)$$

$$= \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{L}x \; e^{-\alpha^2 n^2\pi^2 t/L^2}$$

] This automatically satisfies the B.C.'s $X(0) = X(L) = 0$.

The initial condition now means that

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{L}x$$

Goal: Determine a_n such that the above expression is true.

f could be an arbitrary function on $[0,L]$ so long as $f(0) = f(L) = 0$ in this case. Can arbitrary functions be written as infinite linear comb. of sinusoids? (or cosines, or $e^{in\pi/L x}$'s?)

$\Rightarrow$ Fourier series