December 11, 2019

Last time:

Heat Equation:

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

Also must specify initial condition: $u(x,0) = f(x)$

and boundary condition: $u(0,t) = u(L,t) = 0$

Separation of variables solution:

Assume $u(x,t) = X(x)\,T(t)$, insert into equation:

$$\frac{T'}{\alpha^2 T} = \underbrace{-\lambda}_{\text{constant}} = \frac{X''}{X} \qquad (*)$$

This results in the boundary value problem:

$$\left.\begin{aligned} &X'' + \lambda X = 0 \\ &X(0) = X(L) = 0 \end{aligned}\right\} \text{Non-trivial solutions obtained when } \lambda_n = \frac{n^2\pi^2}{L^2}, \quad X_n(x) = \sin\frac{n\pi}{L}x$$

The equation for T is then:

$$T' + \alpha^2 \lambda T = 0 \qquad \leftarrow \text{use } \lambda_n \text{ in here to obtain:}$$

$$\Rightarrow \quad T_n(t) = e^{-\alpha^2 n^2 \pi^2 t/L^2}$$

Define $u_n(x,t) = X_n(x)\,T_n(t) = \sin\frac{n\pi}{L}x\; e^{-\alpha^2 n^2 \pi^2 t/L^2}$

In $(*)$, λ is allowed to be any of the λ_n's, and therefore u is an arbitrary linear combination:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{L}x\; e^{-\alpha^2 n^2 \pi^2 t/L^2}$$

The initial condition then implies that:

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{L}x$$

Fourier Series

Question: Can an arbitrary function f on the interval $[-L,L]$ be written as:

Note change of interval

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi}{L}x + \sum_{n=0}^{\infty} b_n \sin\frac{n\pi}{L}x \quad ? \qquad (\cancel{**})$$

Note: The functions $\cos\frac{m\pi}{L}x$ and $\sin\frac{n\pi}{L}x$ are orthogonal on the interval $[-L,L]$:

$$\left(\cos\frac{m\pi}{L}x, \sin\frac{n\pi}{L}x\right) = \int_{-L}^{L} \cos\frac{m\pi}{L}x \sin\frac{n\pi}{L}x \, dx$$

$$= 0 \quad (\text{show on your own}).$$

Furthermore, if $m \neq n$, then

$$\int_{-L}^{L} \cos\frac{m\pi}{L}x \cos\frac{n\pi}{L}x \, dx = 0$$

$$\int_{-L}^{L} \sin\frac{m\pi}{L}x \sin\frac{n\pi}{L}x \, dx = 0$$

So the set of functions $\left\{\cos\frac{m\pi}{L}x, \sin\frac{n\pi}{L}x\right\}$ are mutually orthogonal.

Therefore: Assume that f can be written as in $(*)$, then multiplying by these functions and integrating allows us to solve for the a_n's and b_n's:

$$a_n = \frac{1}{\|\cos\frac{n\pi}{L}x\|^2}\left(f(x), \cos\frac{n\pi}{L}x\right)$$

$$= \frac{1}{\underbrace{\int_{-L}^{L}\left(\cos\frac{n\pi}{L}x\right)^2 dx}_{= \begin{cases} L & \text{if } n \neq 0 \\ 2L & \text{if } n = 0 \end{cases}}} \int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x \, dx \quad , n \geq 0$$

And likewise:

$$b_n = \frac{1}{\int_{-L}^{L} \left(\sin \frac{n\pi}{L} x\right)^2 dx} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x \, dx \quad , \quad n \geq 1 .$$

This is merely solving for a_n, b_n's, (by taking inner products, i.e. projections).

Complex version:

Since $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$, $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

We can also expand:

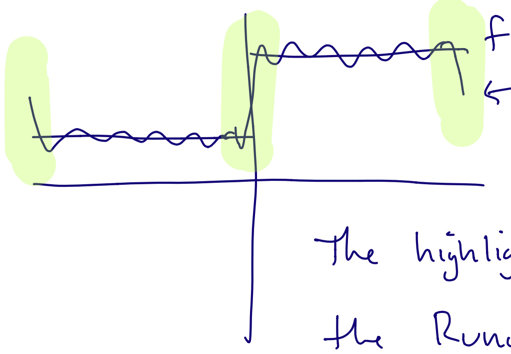
$$f(x) = \sum_{-\infty}^{\infty} c_n e^{in\pi/L \, x} \quad , \quad \text{and} \qquad (**)$$

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-in\pi x/L} .$$

Do these series converge?

Thm: Let f, f' be piecewise continuous on $[-L, L]$. Then, the series (**) and (**) converge to $f(x)$ if f is continuous, and to $\frac{1}{2}\left(f(x^-) + f(x^+)\right)$ if f is discontinuous at x. At $x = \pm L$, the series converges to $\frac{1}{2}\left(f(-L) + f(L)\right)$.

Graphically:



Finite Fourier series approximation to step function

The highlighted portion is known as the Runge Effect (and happens at discontinuity.)

Example: $f(x) = \begin{cases} 0, & x \in [-1,0) \\ 1, & x \in [0,1] \end{cases}$

Then $$a_n = \int_{-1}^{1} f(x) \cos n\pi x \, dx$$

$$= \int_0^1 \cos n\pi x \, dx = \begin{cases} 1 & \text{if } n = 0 \\ \left.\dfrac{\sin n\pi x}{n\pi}\right|_0^1 = \dfrac{\sin n\pi}{n\pi} = 0 & \text{otherwise} \end{cases}$$

$$b_n = \int_{-1}^{1} f(x) \sin n\pi x \, dx$$

$$= \int_0^1 \sin n\pi x \, dx = \left.-\frac{\cos n\pi x}{n\pi}\right|_0^1 = \frac{-\cos n\pi}{n\pi} + \frac{1}{n\pi}$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

So the Fourier series is:

$$f(x) = \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2 \sin (2n+1)\pi x}{(2n+1)\pi}$$

Note that at points of discontinuity $(-1, 0, 1)$ we have that

$$f(-1) = \frac{1}{2} + \sum_{n=0}^{\infty} \underbrace{\frac{2 \sin (2n+1)\pi}{(2n+1)\pi}}_{0} = \frac{1}{2}$$

and likewise $f(0) = f(1) = \frac{1}{2}$.

Ex 2: $f(x) = \begin{cases} 1 & x \in [-2,0) \\ x & x \in [0,2] \end{cases}$

Ex 3: $f(x) = \cos^2 x$ on $[-\pi,\pi]$

Even & Odd Functions

Recall: Even function: $f(-x) = f(x)$

Odd function: $f(-x) = -f(x)$

Also $\cos \frac{n\pi}{L}x$ is EVEN

$\sin \frac{n\pi}{L}x$ is ODD

Therefore, if f is even, its Fourier series only contains cos's

if f is odd, its Fourier series only contains sin's

Since $ODD(x) \cdot EVEN(x) = ODD(x)$

$ODD(x) \cdot ODD(x) = EVEN(x)$

$EVEN(x) \cdot EVEN(x) = EVEN(x)$

Also, if f is odd, then

$$\int_{-L}^{L} f(x)\,dx = \int_{-L}^{0} f(x)\,dx + \int_{0}^{L} f(x)\,dx$$

$$= \int_{0}^{L} -f(x)\,dx + \int_{0}^{L} f(x)\,dx$$

$$= 0$$

Example: If f is odd:

Then $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = 0$ since $f \cdot \cos$ is odd.

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left(\int_{-L}^{0} f(x)\sin\frac{n\pi x}{L}\,dx + \int_{0}^{L} f(x)\sin\frac{n\pi x}{2}\,dx\right)$$

$$= \frac{1}{L}\left(\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx + \int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx\right) = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

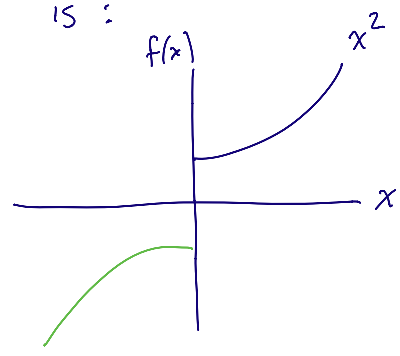
Extensions

Any function on $[0,L]$ can be expanded as either a sine or cosine series by computing an extension.

Let f be defined on $(0,L]$.

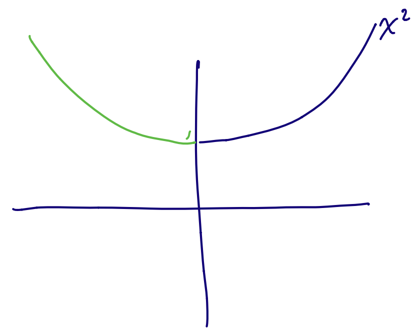
Then the odd extension of f to $[-L,0)$ is:

$$F(x) = \begin{cases} f(x), & x \in [0,L] \\ -f(-x), & x \in [-L,0) \end{cases}$$



The even extension of f to $[-L,0)$ is:

$$G(x) = \begin{cases} f(x), & x \in [0,L] \\ f(-x), & x \in [-L,0) \end{cases}$$



$\Rightarrow$ Since F is odd, it can be expanded in a sine series, since G is even, it can be expanded in a cosine series.

And on $[0,L]$, $F(x) = G(x)$.