Integral Equations: Continuous Theory

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10.1 Reducing the dimension of the computational domain

- Our model problem will be the Laplace equation with Dirichlet data:

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega \\ u = f, & \text{on } \Gamma := \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^2$ is simply connected, open, with smooth boundary Γ.

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  where $\Omega \subset \mathbb{R}^2$ is simply connected, open, with smooth boundary Γ.
- We want a solution u of the form

$$u(\boldsymbol{x}) = \int_\Gamma \phi(\boldsymbol{x} - \boldsymbol{y})\sigma(\boldsymbol{y})\, ds(\boldsymbol{y}), \quad \boldsymbol{x} \in \Omega$$

  where $\phi(\boldsymbol{x}) = -\frac{1}{2\pi}\log(|\boldsymbol{x}|)$ is the free space Green's function for $-\Delta$ in 2D.
- This expression for u looks like a superposition of ϕ weighted by σ. So we formally expect $-\Delta u = 0$ since ϕ is harmonic away from 0.

10.1 Reducing the dimension of the computational domain

- To match the boundary condition, we solve the *Boundary Integral Equation* (BIE) formulation of our original problem:

$$\int_{\Gamma} \phi(\boldsymbol{x} - \boldsymbol{y})\sigma(\boldsymbol{y})\, ds(\boldsymbol{y}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma \tag{1}$$

- From a numerics standpoint, this formulation requires *fewer degrees of freedom* since discretizing Γ is much easier than discretizing Ω.

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- From a numerics standpoint, this formulation requires *fewer degrees of freedom* since discretizing Γ is much easier than discretizing Ω.
- Our *single-layer operator* S is:

$$[S\sigma](\boldsymbol{x}) = \int_\Gamma \phi(\boldsymbol{x}-\boldsymbol{y})\sigma(\boldsymbol{x})\, ds(\boldsymbol{y}) = \int_\Gamma -\frac{1}{2\pi}\log(|\boldsymbol{x}-\boldsymbol{y}|)\sigma(y)\, ds(\boldsymbol{y})$$

- Existence and uniqueness of solutions σ to (1) require some technical assumptions (primarily $f \in C^{1,\alpha}(\Gamma)$ + geometric condition on Ω)[1], but formal manipulations typically hold.

[1] more details in chapters 6, 7 of *Linear Integral Equations* by R. Kress

10.2 Obtaining a well-conditioned mathematical equation

- The BIE (1) leads to linear systems with condition number $O(h^{-1})$ using a grid size h. This beats $O(h^{-2})$ from FD or FEM discretizations.
- The approach in this section will give a BIE leading to condition number *converging* to a finite number as $h \to 0$.

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- The approach in this section will give a BIE leading to condition number *converging* to a finite number as $h \to 0$.
- For $\boldsymbol{y} \in \Gamma$, define

$$d(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{n}(\boldsymbol{y}) \cdot \nabla_{\boldsymbol{y}} \phi(\boldsymbol{x} - \boldsymbol{y}) = \frac{\boldsymbol{n}(\boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y})}{2\pi |\boldsymbol{x} - \boldsymbol{y}|^2}$$

for $\boldsymbol{x} \in \Omega$. This is just the normal derivative of ϕ at $\boldsymbol{y}$.
- Now seek solutions of the form

$$u(\boldsymbol{x}) = \int_\Gamma d(\boldsymbol{x}, \boldsymbol{y}) \sigma(\boldsymbol{y}) \, ds(\boldsymbol{y})$$

10.2 Obtaining a well-conditioned mathematical equation

- Just like before, we expect u to satisfy $-\Delta u = 0$ in Ω, so we just need to worry about matching the boundary condition.
- The singularity from $d(\boldsymbol{x}, \boldsymbol{y})$ is stronger than from $\phi(\boldsymbol{x} - \boldsymbol{y})$. Turns out $[S\sigma](\boldsymbol{x})$ is continuous as you approach Γ, but when using $d(\boldsymbol{x}, \boldsymbol{y})$ we pick up an extra term.

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- Our BIE formulation now becomes

$$-\frac{1}{2}\sigma(\boldsymbol{x}) + \int_\Gamma d(\boldsymbol{x}, \boldsymbol{y})\sigma(\boldsymbol{y})\, ds(\boldsymbol{y}) = f(\boldsymbol{x}), \quad x \in \Gamma \tag{2}$$

- Our *double-layer operator D* is

$$[D\sigma](\boldsymbol{x}) = \int_\Gamma d(\boldsymbol{x}, \boldsymbol{y})\sigma(\boldsymbol{y})\, ds(\boldsymbol{y}) = \int_\Gamma \frac{\boldsymbol{n}(\boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y})}{2\pi|\boldsymbol{x} - \boldsymbol{y}|^2}\sigma(\boldsymbol{y})\, ds(\boldsymbol{y})$$

Note $[D\sigma]$ is defined on $\overline{\Omega}$, but has a jump as you approach Γ.

10.2 Obtaining a well-conditioned mathematical equation

- The BIE $\left(-\frac{1}{2}I + D\right)\sigma = f$ is a Fredholm equation of the second kind, and technical results regarding compact operators tell us that discretizations of this BIE lead to exceedingly well-conditioning systems.
- Even better, the eigenvalues for discretizations of $\left(-\frac{1}{2}I + D\right)\sigma = f$ are clustered near $-1/2$, so we can expect iterative solvers to converge rapidly (with # of iterations independent of grid size)

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10.3 External domain and B.C. at ∞ for Laplace equation

- What about exterior problems?

$$\begin{cases} -\Delta u = 0, & \text{in } \Omega \\ u = f, & \text{on } \Gamma \\ \lim_{|\boldsymbol{x}|\to\infty}\left(u(\boldsymbol{x}) + \frac{Q}{2\pi}\log|\boldsymbol{x}|\right) = 0, & \text{for some } Q \in \mathbb{R} \end{cases}$$

  where now Ω is the domain *exterior* to the smooth close contour Γ. The third line is a growth condition at ∞.
- The computational domain Ω is *unbounded*, so if we used FD or FEM methods, we would have to artificially truncate the domain and impose artificial boundary conditions.

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- The computational domain Ω is *unbounded*, so if we used FD or FEM methods, we would have to artificially truncate the domain and impose artificial boundary conditions.
- Conversion to a BIE makes the computational domain Γ which is *bounded*. With a single-layer potential, the solution is $u(\boldsymbol{x}) = [S\sigma](\boldsymbol{x})$ in Ω, where $\sigma(\boldsymbol{y})$ solves the BIE:

$$[S\sigma](\boldsymbol{x}) = \int_\Gamma \phi(\boldsymbol{x} - \boldsymbol{y})\sigma(\boldsymbol{y})\, ds(\boldsymbol{y}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma \tag{3}$$

10.3 External domain and B.C. at ∞ for Laplace equation

- The single-layer solution u automatically satisfies $-\Delta u = 0$ in Ω like before, and also automatically satisfies the growth condition since

$$\inf_{y \in \Gamma} |\phi(\boldsymbol{x} - \boldsymbol{y})| \lesssim |u(\boldsymbol{x})| \lesssim \sup_{y \in \Gamma} |\phi(\boldsymbol{x} - \boldsymbol{y})|, \quad \forall \boldsymbol{x} \in \Omega$$

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- Similar to the interior Dirichlet problem, the single layer formulation upon discretization gives linear systems whose condition number grows with $\#$ of points describing Γ.
- If we want to use a double-layer formulation, we need to correct for the growth condition, since

$$[D\sigma](\boldsymbol{x}) = \int_\Gamma d(\boldsymbol{x}, \boldsymbol{y}) \sigma(\boldsymbol{y}) \, ds(\boldsymbol{y}) = \int_\Gamma \frac{\boldsymbol{n}(\boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y})}{2\pi |\boldsymbol{x} - \boldsymbol{y}|^2} \sigma(\boldsymbol{y}) \, ds(\boldsymbol{y})$$

should now decay like $|\boldsymbol{x}|^{-1}$.

- To manually correct for the decay of the double-layer, we fix z interior to Γ, and look for solutions of the form

$$u(\boldsymbol{x}) = [D\sigma](\boldsymbol{x}) + \phi(\boldsymbol{x} - \boldsymbol{z}) \int_\Gamma \sigma(\boldsymbol{y})\, ds(\boldsymbol{y})$$

These solutions now satisfy the growth condition and still satisfy $-\Delta u = 0$ in Ω since $\boldsymbol{z} \notin \Omega$. The resulting BIE for σ is then

$$\frac{1}{2}\sigma(\boldsymbol{x}) + \int_\Gamma [d(\boldsymbol{x},\boldsymbol{y}) + \phi(\boldsymbol{x} - \boldsymbol{z})]\, \sigma(\boldsymbol{y})\, ds(\boldsymbol{y}) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in \Gamma \qquad (4)$$

note that in the exterior problem we pick up a term $+\frac{1}{2}\sigma(x)$ while in the interior problem we picked up $-\frac{1}{2}\sigma(x)$.

10.4 The Helmholtz equation

- Other PDE can be also be solved using a BIE formulation. Consider the interior Dirichlet problem for the Helmholtz equation with positive wave number κ.

$$\begin{cases} -\Delta u - \kappa^2 u = 0, & \text{in } \Omega \\ u = f, & \text{on } \Gamma \end{cases}$$

- The free space Green's function for the Helmholtz operator is given by the zeroth order Hankel function: $\phi_\kappa(\boldsymbol{x}) = \frac{i}{4} H_0^{(1)}(\kappa|\boldsymbol{x}|)$.

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- We can repeat the exact same process as for $-\Delta$ and get the single and double-layer operators:

$$[S_\kappa \sigma](\boldsymbol{x}) = \int_\Gamma \phi_\kappa(\boldsymbol{x} - \boldsymbol{y})\sigma(\boldsymbol{y})\, ds(\boldsymbol{y})$$

$$[D_\kappa \sigma](\boldsymbol{x}) = \int_\Gamma d_\kappa(\boldsymbol{x}, \boldsymbol{y})\sigma(\boldsymbol{y})\, ds(\boldsymbol{y})$$

where $d_\kappa(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{n}(\boldsymbol{y}) \cdot \nabla_{\boldsymbol{y}} \phi_\kappa(\boldsymbol{x} - \boldsymbol{y})$.

10.4 The Helmholtz equation

- The function $\phi_\kappa(\boldsymbol{x})$ has a log-singularity near the origin, just like $\phi(\boldsymbol{x})$, hence we expect the layer operators to behave similarly.
- If we try to use a double-layer formulation and look for solutions of the form $u(\boldsymbol{x}) = [D_\kappa\sigma](\boldsymbol{x})$, we get the BIE $\left(-\frac{1}{2}I + D_\kappa\right)\sigma = f$ on Γ which is not well defined for all κ.

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- To remedy this, the *combined field* formulation uses a linear combination of S_κ and D_κ. We look for solutions of the form $u(\boldsymbol{x}) = [(D_\kappa + i\eta S_\kappa)\sigma](\boldsymbol{x})$ where $\eta = \pm\kappa$. The resulting BIE is

$$\left[\left(-\frac{1}{2}I + D_\kappa + i\eta S_\kappa\right)\sigma\right](\boldsymbol{x}) = f(\boldsymbol{x}), \quad x \in \Gamma \tag{5}$$

10.5 Radiation conditions for Helmholtz equation

- Just like for the Laplace equation, we can also apply a BIE formulation for exterior Helmholtz problems:

$$\begin{cases} -\Delta u - \kappa^2 u = 0, & \text{in } \Omega \\ u = f, & \text{on } \Gamma \\ \lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u(r\boldsymbol{z})}{\partial r} - i\kappa u(r\boldsymbol{z})\right) = 0, & \text{for every unit vector } \boldsymbol{z} \end{cases}$$

  where the last term is a condition at ∞. This exterior equation is useful in modeling certain types of scattering problems.

- Using the combined field formulation and guessing solutions like $u(\boldsymbol{x}) = [(D_\kappa + i\eta S_\kappa)\sigma](\boldsymbol{x})$, the corresponding BIE for σ is

$$\left[\left(\frac{1}{2}I + D_\kappa + i\eta S_\kappa\right)\sigma\right](\boldsymbol{x}) = f(\boldsymbol{x}), \quad x \in \Gamma \tag{6}$$

10.6 "Direct" derivation of BIE for harmonic potentials

- Here, we derive a direct method of reformulating Laplace's equation as a BIE. Let $s(\boldsymbol{x}, \boldsymbol{y}) = \phi(\boldsymbol{x} - \boldsymbol{y})$ and $d(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{n}(\boldsymbol{y}) \cdot \nabla_{\boldsymbol{y}} \phi(\boldsymbol{x} - \boldsymbol{y})$.

Theorem

Let Γ be a smooth, bounded domain in $\mathbb{R}^2$. For any u such that $-\Delta u = 0$ in Ω, then for $\boldsymbol{x} \in \mathbb{R}^2$:

$$\theta(\boldsymbol{x}) u(\boldsymbol{x}) = \int_\Gamma \left(s(\boldsymbol{x}, \boldsymbol{y}) \frac{\partial u(\boldsymbol{y})}{\partial \boldsymbol{n}} - d(\boldsymbol{x}, \boldsymbol{y}) u(\boldsymbol{y}) \right) ds(\boldsymbol{y}), \tag{7}$$

where

$$\theta(\boldsymbol{x}) = \begin{cases} 1 & \text{for } \boldsymbol{x} \in \Omega \\ 1/2 & \text{for } \boldsymbol{x} \in \Gamma \\ 0 & \text{for } \boldsymbol{x} \in \overline{\Omega}^c \end{cases}$$

10.6 "Direct" derivation of BIE for harmonic potentials

- Given boundary conditions on Γ, we can use (7) to immediately convert the PDE to a BIE:
  - *Dirichlet data*: $u = f$ on Γ. Then (7) gives the BIE for $\frac{\partial u}{\partial \boldsymbol{n}}\big|_\Gamma$:

$$\int_\Gamma s(\boldsymbol{x}, \boldsymbol{y}) \frac{\partial u(\boldsymbol{y})}{\partial \boldsymbol{n}} \, ds(\boldsymbol{y}) = \frac{1}{2} f(\boldsymbol{x}) + \int_\Gamma d(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{y}) \, ds(\boldsymbol{y}), \quad \boldsymbol{x} \in \Gamma$$

  If we solve this BIE for $\frac{\partial u}{\partial \boldsymbol{n}}\big|_\Gamma$, we can use (7) to recover u in Ω.

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    If we solve this BIE for $\frac{\partial u}{\partial \boldsymbol{n}}\big|_\Gamma$, we can use (7) to recover u in Ω.
  - *Neumann data*: $\frac{\partial u}{\partial \boldsymbol{n}} = f$ on Γ. Then (7) gives the BIE for $u\big|_\Gamma$:

$$\frac{1}{2} u(\boldsymbol{x}) + \int_\Gamma d(\boldsymbol{x},\boldsymbol{y}) u(\boldsymbol{y})\, ds(\boldsymbol{y}) = \int_\Gamma s(\boldsymbol{x},\boldsymbol{y}) f(\boldsymbol{y})\, ds(\boldsymbol{y}), \quad \boldsymbol{x} \in \Gamma$$

    If we solve this BIE for $u\big|_\Gamma$, we can use (7) to recover u on Ω.

10.6 "Direct" derivation of BIE for harmonic potentials

- Equation (7) also tells us why we pick up a factor of $\frac{1}{2}$ in the double layer formulation. Applying (7) with $u \equiv 1$ gives

$$\int_\Gamma d(\boldsymbol{x}, \boldsymbol{y}) \, ds(\boldsymbol{y}) = \begin{cases} -1, & \text{for } \boldsymbol{x} \in \Omega \\ -1/2, & \text{for } \boldsymbol{x} \in \Gamma \\ 0, & \text{for } \boldsymbol{x} \in \overline{\Omega}^c \end{cases}$$

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- Then for continuous σ defined on Γ and $\boldsymbol{x} \in \Gamma$, we get

$$\lim_{\boldsymbol{x}' \to \boldsymbol{x}} [D\sigma](\boldsymbol{x}') = -\frac{1}{2}\sigma(\boldsymbol{x}) + [D\sigma](\boldsymbol{x}), \quad \boldsymbol{x}' \in \Omega$$

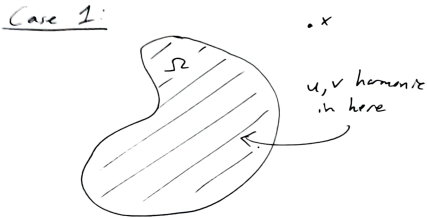
- Proof sketch: $[D\sigma](\boldsymbol{x}') = \int_\Gamma d(\boldsymbol{x}', \boldsymbol{y})(\sigma(\boldsymbol{y}) - \sigma(\boldsymbol{x}))\, ds(\boldsymbol{y}) - \sigma(\boldsymbol{x})$, assume some regularity on σ, swap limit with integral.

10.6 "Direct" derivation of BIE for harmonic potentials

- Proof outline for Theorem. For a fixed $\boldsymbol{x} \in \mathbb{R}^2$, set $v(\boldsymbol{y}) = \phi(\boldsymbol{x} - \boldsymbol{y})$. Green's 2nd identity says

$$\int_\Omega u\Delta v - v\Delta u = \int_\Gamma d(\boldsymbol{x}, \boldsymbol{y})u(\boldsymbol{y}) - s(\boldsymbol{x}, \boldsymbol{y})\frac{\partial u(\boldsymbol{y})}{\partial \boldsymbol{n}}\, ds(\boldsymbol{y}) \qquad (8)$$

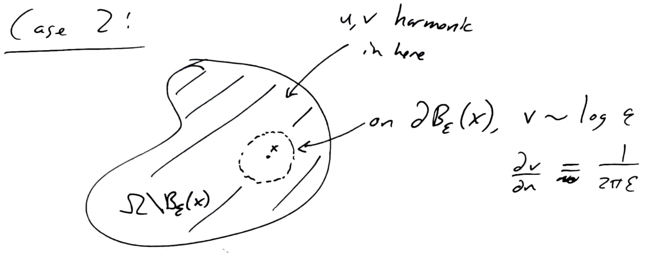
  - *Case 1*: $\boldsymbol{x} \in \overline{\Omega}^c$. Then u, v harmonic in Ω, so LHS of (8) is 0.



10.6 "Direct" derivation of BIE for harmonic potentials

- *Case 2*: $\boldsymbol{x} \in \Omega$. Now v is not harmonic in Ω. Let $B_\varepsilon(\boldsymbol{x})$ be ball of radius ε centered at $\boldsymbol{x}$. Apply (8) to $\Omega \backslash B_\varepsilon(\boldsymbol{x})$ and show

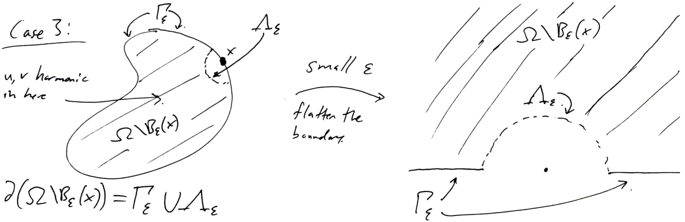
$$\lim_{\varepsilon \to 0^+} \int_{\partial B_\varepsilon(\boldsymbol{x})} u \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} = u(\boldsymbol{x}), \quad \lim_{\varepsilon \to 0^+} \int_{\partial B_\varepsilon(\boldsymbol{x})} v \frac{\partial u}{\partial \boldsymbol{n}} = 0$$



10.6 "Direct" derivation of BIE for harmonic potentials

- *Case 3*: $\boldsymbol{x} \in \Gamma$. Nearly same argument as in Case 2, but the cut boundary is slightly different. Apply (8) to $\Omega \backslash B_\varepsilon(\boldsymbol{x})$

$$\lim_{\varepsilon \to 0^+} \int_{\Lambda_\varepsilon} u \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} = \frac{1}{2} u(\boldsymbol{x}), \quad \lim_{\varepsilon \to 0^+} \int_{\Lambda_\varepsilon} v \frac{\partial u}{\partial \boldsymbol{n}} = 0$$



locally, Λ_ε looks like a semicircle with radius ε, on Λ_ε, $v \sim \log \varepsilon$, $\frac{\partial v}{\partial n} \approx \frac{1}{\pi \varepsilon}$

11.1 Problems with body loads

- Now consider a body load g for Laplace's equation:

$$\begin{cases} -\Delta u = g, & \text{in } \Omega \\ u = f, & \text{on } \Gamma \end{cases}$$

- We can first compute a particular solution u_p which satifies $-\Delta u_p = g$ on Ω, *ignoring boundary conditions*. Analytically: $u_p(\boldsymbol{x}) = \int_\Omega \phi(\boldsymbol{x} - \boldsymbol{y})g(\boldsymbol{y})\, d\boldsymbol{y}$ which will satisfy $-\Delta u = g$ in Ω.
- Then set $u_h = u - u_p$ which solves (via BIE formulation):

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- Then set $u_h = u - u_p$ which solves (via BIE formulation):

$$\begin{cases} -\Delta u_h = 0, & \text{in } \Omega \\ u_h = f - u_p, & \text{on } \Gamma \end{cases}$$

- Computing u_p can be challenging, due complicated Ω and singular ϕ. That said, there are methods of extending Ω and g to be simpler computationally (e.g. put Ω inside a big box and smoothly extend g). Then specialized methods like FMM or FFT can evaluate u_p fast.

11.2 Variable coefficient PDE; Lippmann-Schwinger equation

- For variable coefficient PDE, integral formulations are still possible but typically they are *volume integral equations*.
- While these formulations lose the benefit of reducing the dimension of the computational domain, they still retains the benefits of finite computational domain + well-conditioned systems.

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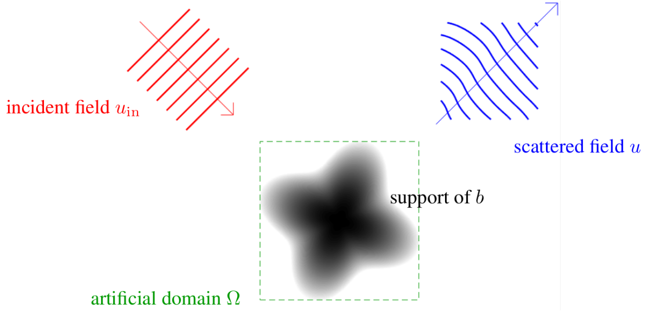
- For variable coefficient PDE, integral formulations are still possible but typically they are *volume integral equations*.
- While these formulations lose the benefit of reducing the dimension of the computational domain, they still retains the benefits of finite computational domain + well-conditioned systems.
- As an example, consider the free space, variance coefficient Helmholtz equation:

$$\begin{cases} -\Delta u(\boldsymbol{x}) - \kappa^2(1 - b(\boldsymbol{x})^2)u(\boldsymbol{x}) = -\kappa^2 b(\boldsymbol{x}) u_{\text{in}}(\boldsymbol{x}), & \text{in } \mathbb{R}^2 \\ \frac{\partial u(\boldsymbol{x})}{\partial r} - i\kappa u(\boldsymbol{x}) = o(r^{-1/2}), & \text{as } r = |\boldsymbol{x}| \to \infty. \end{cases}$$

which models acoustic wave propagation in a medium with variable wave speed. Assume b is smooth, vanishes outside Ω, and bounded by 1, and that u_{in} solves Helmholtz in Ω with constant κ.

11.2 Variable coefficient PDE; Lippmann-Schwinger equation



$$\begin{cases} -\Delta u(\boldsymbol{x}) - \kappa^2(1-b(\boldsymbol{x})^2)u(\boldsymbol{x}) = -\kappa^2 b(\boldsymbol{x})u_{\text{in}}(\boldsymbol{x}), & \text{in } \mathbb{R}^2 \\ \frac{\partial u(\boldsymbol{x})}{\partial r} - i\kappa u(\boldsymbol{x}) = o(r^{-1/2}), & \text{as } r = |\boldsymbol{x}| \to \infty. \end{cases}$$

- Here, b indicates how much wave speed in Ω differs compared to free space wave speed.

11.2 Variable coefficient PDE; Lippmann-Schwinger equation

- Free space Green's function for Helmwoltz with radiating BC is $G_\kappa(\boldsymbol{x}, \boldsymbol{y}) = \frac{i}{4} H_0^{(1)}(|\boldsymbol{x} - \boldsymbol{y}|)$, where $H_0^{(1)}$ is the zeroth order Hankel function.
- Search for solutions of the form

$$u(\boldsymbol{x}) = \int_\Omega G_\kappa(\boldsymbol{x}, \boldsymbol{y})\sigma(\boldsymbol{y})\, d\boldsymbol{y}, \quad \boldsymbol{x} \in \mathbb{R}^2$$

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- This leads to the BIE for σ:

$$\sigma(\boldsymbol{x}) + \kappa^2 b(\boldsymbol{x})\int_\Omega G_\kappa(\boldsymbol{x},\boldsymbol{y})\sigma(\boldsymbol{y})\,d\boldsymbol{y} = -\kappa^2 b(\boldsymbol{x})u_{\text{in}}(\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega$$

- Computational domain is now Ω which is bounded (instead of $\mathbb{R}^2$), and the above BIE leads to well-conditioned systems (just like double-layer formulation).

Summary + Extensions

- Integral equations serve as a powerful, alternative modeling tool to PDE. Benefits include
  - Reducing dimension of computational domain (Ω down to Γ).
  - Well-conditioned systems upon discretization (e.g. double-layer formulation)
  - Can handle exterior problems with a *finite* computational domain.
- Different BIE formulations with different properties can be found for the same PDE.

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  - Well-conditioned systems upon discretization (e.g. double-layer formulation)
  - Can handle exterior problems with a *finite* computational domain.
- Different BIE formulations with different properties can be found for the same PDE.
- With extra work/challenges, can be extended to other types of models (e.g. linear elasticity, Stokes flow, time-Harmonic Maxwell).
- 3D is possible, but Γ harder to treat as a surface + kernels are more singular.